PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES
Volume 21 May 15, 1935 Number 5

RATE OF GROWTH AND LENGTH OF LIFE

By H. C. Sherman and H. L. Campbell

Department of Chemistry, Columbia University

Read before the Academy, Tuesday, November 20, 1934

Previously we have recorded[1,2,3] experiments in which, starting with a diet already adequate to the maintenance of normal nutrition and health for generation after generation, an improvement of this diet, by better quantitative adjustment of the natural foods of which it is composed, results in an increased rate and efficiency of growth, a higher level and longer period of adult vitality, and an extension of the average length of life and of the life-expectation of the adult.

More recently, McCay and Crowell,[4] in a different type of experiment, find that restricted as contrasted with maximal growth may also be followed by longer life.

Under what conditions, then, may an increased rate of growth be expected to be followed by longer life as in our previous work, when may an inverse relationship such as appears in McCay and Crowell's experiments be expected, and when do rate of growth and length of life vary independently?

The present paper deals with the differences observed within groups which were homogeneous in the sense of involving no known or experimentally controllable variable. In other words, it deals with the individual variations among laboratory rats of initial good health, of the same heredity, living in the same environment, and subsisting upon one or the other, respectively, of the two normal diets described in our previous papers as Diet A (which was adequate in the sense explained) and Diet B (which proved to be better). Animals on these two diets are here treated separately, and the records of the two sexes are separate; so that we are here dealing with differences which are not attributable to any other variable than such as, on the one hand, individual "growth impulse" and on the other hand, individual "constitution predisposing to longer life." Do these vary interdependently, or independently?

Plainly this is a different problem from that presented by experimentally

induced nutritional differences; but one which bears upon the interpretation of the findings of the food experiments.

TABLE 1

LENGTH OF LIFE ON RATE OF GROWTH: HALVES OF "HOMOGENEOUS" GROUPS. AVERAGES WITH THEIR PROBABLE ERRORS

	MALES		FEMALES	
	DIET A	DIET B	DIET A	DIET B
Number of cases	135	124	196	163
Gain during 5th–8th weeks of life (inclusive)				
Slower-growing half—Gm.	43.9 ± 1.00	76.0 ± 1.24	39.0 ± 0.93	60.6 ± 0.65
Faster-growing half—Gm.	79.8 ± 1.08	107.0 ± 0.88	67.5 ± 0.42	79.8 ± 0.41
Difference with its P. E.	35.9 ± 1.47	31.0 ± 1.52	28.5 ± 1.02	19.2 ± 0.77
Ratio of difference to its P. E.	24.4	20.4	27.9	24.9
Average age at natural death				
Slower-growing half—Days	576 ± 12.5	618 ± 13.6	591 ± 12.9	677 ± 11.3
Faster-growing half—Days	567 ± 9.7	646 ± 10.3	616 ± 8.9	664 ± 10.7
Difference with its P. E.	−9 ± 15.8	+28 ± 17.0	+25 ± 15.7	−13 ± 15.5
Ratio of difference to its P. E.	−0.6	+1.6	+1.6	−0.8

Table 1 compares the average lengths of life of the slower-growing and faster-growing halves of each of the four "homogeneous" groups, i.e., of each sex on each of the two diets. We have recently shown[5] that the usual statistical interpretation is applicable with a high degree of accuracy to growth data of this type. Hence all four of the differences in rate of growth with which Table 1 deals are unquestionably significant as differences. Yet no relationship to the length of life subsequently attained by the same individuals appears; for in two of the four cases the greater average length of life was shown by the slower-growing, and in the other two by the faster-growing, half. In none of these four cases has the difference in rate of growth been followed by a difference in length of life which is statistically significant in the accepted interpretation of the term. And if we combine the findings of the four comparisons, taking account of sign, the apparent lack of relationship is strongly confirmed.

Further confirmation is afforded by comparison of the slowest-growing with the fastest-growing quarter of each of the four groups—the data of this comparison being here omitted in the interest of brevity.

Inasmuch as we have stronger evidence of the quantitative validity of the usual statistical interpretation as applied to rate of growth than as

applied to length of life, the evidence of these findings becomes more strictly quantitative, and in this respect more convincing, when each of the same four "homogeneous" groups is divided into a shorter-lived and a longer-lived half and these then compared as to the rates at which they had grown. Table 2 summarizes the data as thus arranged. In only one of the four comparisons would the difference in average rate of growth as between the longer-lived and shorter-lived of comparable individuals appear statistically significant if it stood alone. When account is taken of the four parallel findings with respect to the sign as well as to the probable error of each difference it appears that the difference as indicated by this experiment as a whole is only about one-fifth of its probable error and hence that the chances are more than 1000 to 1 that the variations in rate of growth and length of life are independent rather than interdependent.

TABLE 2

RATE OF GROWTH ON LENGTH OF LIFE: HALVES OF "HOMOGENEOUS" GROUPS. AVERAGES WITH THEIR PROBABLE ERRORS

	MALES		FEMALES	
	DIET A	DIET B	DIET A	DIET B
Number of cases	135	124	196	163
Age of shorter-lived half—Days	469 ± 9.2	532 ± 10.5	475 ± 8.5	555 ± 8.1
Age of longer-lived half—Days	665 ± 5.5	735 ± 5.7	732 ± 5.5	786 ± 5.5
Difference with its P. E.	196 ± 10.7	203 ± 11.9	257 ± 10.1	231 ± 9.8
Ratio of difference to its P. E.	18.3	17.1	25.4	23.5
Growth in grams during 5th–8th weeks (incl.) of life				
Shorter-lived half	66.0 ± 1.91	89.8 ± 1.85	51.8 ± 1.26	69.4 ± 1.02
Longer-lived half	57.8 ± 1.58	93.1 ± 1.53	54.6 ± 1.14	70.9 ± 0.73
Difference ± P. E.	−8.2 ± 2.48	+3.3 ± 2.4	+2.8 ± 1.70	+1.5 ± 1.25
Ratio of difference to its P. E.	−3.3	+1.4	+1.6	+1.2

If the one case in which the difference which might superficially appear to be significant were more than accidental it should be confirmed and accentuated (and other cases might then also appear) upon comparing the data for the shortest-lived and longest-lived quarters of each of the same four groups. This is done in Table 3; and these findings confirm the more comprehensive view of the data of Table 2, for here none of the differences appears even superficially significant, the difference for the comparison as a whole is again only about one-fifth of its probable error, and (stronger evidence that these differences are only accidental) the *sign* of the difference is here reversed.

TABLE 3

RATE OF GROWTH ON LENGTH OF LIFE: FIRST AND FOURTH QUARTERS OF "HOMOGENEOUS" GROUPS.

AVERAGES WITH THEIR PROBABLE ERRORS

	MALES		FEMALES	
	DIET A	DIET B	DIET A	DIET B
Age of shortest-lived quarter, Days	389 ± 12.7	453 ± 15.6	386 ± 11.7	469 ± 10.0
Age of longest-lived quarter, Days	730 ± 5.7	784 ± 7.1	796 ± 6.5	842 ± 6.9
Difference ± its P. E.	341 ± 13.9	331 ± 17.2	410 ± 13.4	373 ± 12.1
Ratio of difference to its P. E.	24.5	19.2	30.6	30.8
Growth in grams during 5th to 8th weeks (incl.) of life				
Shortest-lived quarter	65.7 ± 3.17	91.4 ± 2.97	50.7 ± 1.80	70.1 ± 1.33
Longest-lived quarter	60.8 ± 2.19	96.1 ± 1.87	49.6 ± 1.87	69.6 ± 1.05
Difference ± its P. E.	−4.9 ± 3.86	+4.7 ± 3.51	−1.1 ± 2.59	−0.5 ± 1.69
Ratio of difference to its P. E.	−1.3	+1.3	−0.4	−0.3

When the longer- and shorter-lived halves or quarters are compared as to their growth in the portion of the life cycle preceding that used in the above comparison, it again appears that among the individuals of such a homogeneous group, rate of growth and length of life vary independently of each other.

An expression of the final outcome of all our comparisons of the longer-lived with the shorter-lived individuals of the same sex on the same diet indicates that the difference, if any, in rate of growth between the longer-lived and the shorter-lived was only about one-fifth its probable error; which, in view of previous statistical study of such growth data, may be interpreted as showing clearly that rate of growth and length of life here vary independently of each other. Or, differently stated, among individuals of the same sex and the same heredity, on the *same normal* diet, those who grew faster and those who grew more slowly had equally good prospects of a long life.

It is *also* evident that certain differences in food may influence *both* the rate of growth and the length of life. Studies of the influence of food upon length of life, with related observations upon growth and health, are being continued with the coöperation of the Carnegie Corporation of New York and the Carnegie Institution of Washington.

[1] H. C. Sherman and H. L. Campbell, *Jour. Biol. Chem.*, **60**, 5–15 (1924).
[2] H. C. Sherman and H. L. Campbell, *Proc. Nat. Acad. Sci.*, **14**, 852–855 (1928).
[3] H. C. Sherman and H. L. Campbell, *Jour. Nutrition*, **2**, 415–417 (1930).

[4] C. M. McCay and M. F. Crowell, *Anat. Rec.*, **57**, No. 4, Suppl. p. 102 (1933); *Nutr. Abs. and Rev.*, **3**, 1119 (1934); *Sci. Mo.*, November (1934).

[5] H. C. Sherman and H. L. Campbell, *Proc. Nat. Acad. Sci.*, **20**, 413–416 (1934).

---

*THE GRAPTOLITES OF THE SIMPSON GROUP OF OKLAHOMA**

BY CHARLES E. DECKER

DEPARTMENT OF PALEONTOLOGY, UNIVERSITY OF OKLAHOMA

Communicated March 27, 1935

The older sedimentary formations of the Arbuckle Mountains of Oklahoma beginning with the oldest are: Reagan sandstone (Cambrian), Arbuckle limestone (Cambro-Ordovician), Simpson group, Viola limestone, Sylvan shale (all three Ordovician), Chimneyhill limestone and Henryhouse shale (both Silurian).

The Simpson group has been divided by the writer[1] into five formations, Joins, Oil Creek, McLish, Tulip Creek and Bromide. All five of these formations are present in the western part of the mountains, but the oldest and next to the youngest (Joins and Tulip Creek) wedge out in the middle part of the mountains, leaving only three formations in the eastern part.

In the formations named above, graptolites have been found in the Arbuckle, Joins, Bromide, Viola, Sylvan and Henryhouse. Considerable progress has been made recently on the study of the graptolites of these formations. They are listed by zones in a paper on the Viola limestone,[2] and thirty species and varieties from this formation have been described and illustrated.[3] Two more papers have been sent to press recently on the graptolites of the Henryhouse and Sylvan shales. Yet much remains to be done on the graptolites of the Arbuckle Mountains and on those in other formations in the eastern part of the State. The study thus far has yielded excellent results in local and in widespread correlation.

In the Simpson group graptolites have been found in only two horizons, one near the base in the lower part of the Joins formation, and one near the top in the upper part of the Bromide formation. The lower zone was found and traced around the mountains by the writer and Rex McGehee in 1928. Earlier Chester A. Reeds had found *Didymograptus artus* in this zone and had sent them to E. O. Ulrich,[4] who placed them provisionally in the upper part of the Arbuckle limestone.

As a rule *Didymograptus artus* was found in a zone only 2 inches wide, but in the Criner Hills it ranges through 8 feet of sediments. The zone was found to vary from 43 to 102 feet above the base of the Joins formation. It was found also 50 feet above the base of the Joins by D. A. McGee at

Oklahoma City at a depth of 6316 feet in the Williamson-Canfield well no. 1, and at 6302 in the Foster well no. 1. The presence of the *Didymograptus artus* zone at Oklahoma City proved that the Joins formation extends into that area, though some geologists had thought earlier that it was not present. Also, this species is very important for wider correlation, as it occurs definitely in the *D. bifidus* zone.[5] An illustration of the latter is shown in plate 1, figures 10, 10*a* for comparison. The specimen illustrated came from shales in the Black Rock limestone near Black Rock, Arkansas. Because of the association of these two species of *Didymograptus*, the lower part of the Joins formation may be connected with the widespread *D. bifidus* zone. In this country outside of Arkansas, this zone occurs in the Quebec shales at Point Levis, three miles above the River St. Anne in Quebec.[6] It is common in beds 3 to 5 in the Deep Kill section of eastern New York,[7] and it has been listed from Nevada by Gurley.[8] In Great Britain it occurs in England, Wales, Scotland and Ireland, where it has been found in the middle and upper Arenig beds, but chiefly in the latter.[9] In Europe it occurs in Norway, Sweden, Bohemia and France, also in Victoria, Australia.[10]

Classification of Graptolites

GRAPTOLOIDEA AXONOLIPA (Frech) em Ruedemann
Genus DIDYMOGRAPTUS McCoy 1851

Didymograptus artus Elles and Wood.

GRAPTOLOIDEA AXONOPHORA (Frech)
Genus DIPLOGRAPTUS McCoy 1854

Diplograptus (*Amplexograptus*) *maxwelli n. sp.*

Description of Species

DIDYMOGRAPTUS ARTUS Elles and Wood[11]
Plate 1, figures 8, 9, 9*a*

Didymograptus artus is a tiny form of the pendent type in which the two stipes diverge at the proximal end at an angle of 90°, then the stipes bend inward until they become nearly parallel. The stipes are 0.4 mm. wide at the proximal end, and soon increase to a width of 1.3 mm. A colony is commonly 8 mm. long, but the length may be increased to 3 times that amount. The thecae are very close set, 18 or 19 in 10 mm., and they slope at an angle of 50°. The thecae are long and narrow, about 3 times as long as wide, and they overlap one-half to two-thirds their length. The sicula is about 1.3 mm. long.

Didymograptus artus is much like the cosmopolitan *D. bifidus* in whose

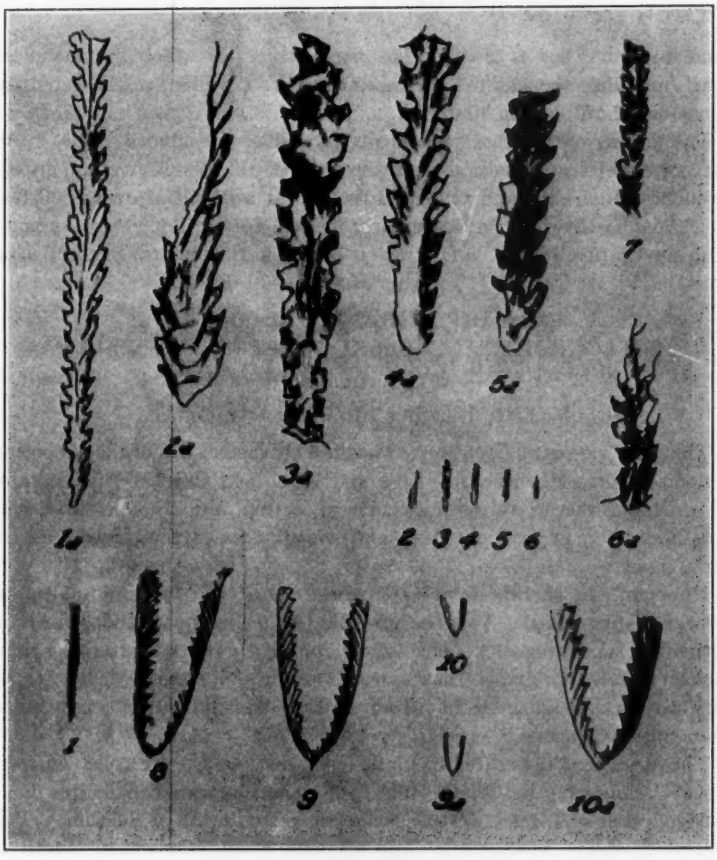

PLATE 1

Diplograptus (Amplexograptus) maxwelli n. sp.

Figures 1 to 7, cotypes nos. A2041 to A2047 are in Univ. of Oklahoma Invert. Paleon. Mus.

Figures 1, 1*a*, 2 miles west of Nebo Store; × 1, × 4.

Figures 2, 2*a* to 6, 6*a*, Rock Crossing, Criner Hills; × 1, × 8.

Figure 7, Rock Crossing, Criner Hills; × 4.

Didymograptus artus Elles and Wood

Figures 8, 9, 9*a*, plesiotypes nos. A2048, A2049, Univ. of Oklahoma Invert. Paleon. Mus.

Figure 8, West side U. S. Highway 77, 3 miles north of Springer; × 4.

Figures 9, 9*a*, Williamson-Canfield no. 1, 6316 feet, Oklahoma City; × 4, × 1.

Didymograptus bifidus Hall

Figures 10, 10*a*, half mile east of Black Rock, Ark.; × 1, × 4.

Illustrations given for comparison.

zone it occurs, but is smaller, has narrower stipes which become more nearly parallel, and has the thecae much more crowded, to the extent of about one-third more in 10 mm.

The finding of *Didymograptus artus* extensively developed in the lower part of the Joins formation near the base of the Simpson group in the Arbuckle Mountains and Criner Hills, and at a depth of over 6300 feet at Oklahoma City, has helped greatly in local correlation. Also, because of its association with *D. bifidus*, it has helped in the long range correlation noted above.

Genus DIPLOGRAPTUS McCoy 1851

DIPLOGRAPTUS (AMPLEXOGRAPTUS) MAXWELLI

Decker n. sp.

Plate 1, figures 1 to 7 and 1*a* to 6*a*

This new species of *Diplograptus* has the characteristics of the subgroup, *Amplexograptus*, described by Elles and Wood as follows:[12] "Mesograpti in which the thecae were semicircular in section, and the apertural margins undulate; excavations deep and conspicuous in the obverse aspect throughout."

No complete rhabdosome was secured, and no synrhabdosome was clearly distinguished. Yet the crowding of so many rhabdosomes together in a small area with some of them apparently radiating from a center, makes it seem probable that synrhabdosomes did exist. They occur generally in a tough limestone from which it was difficult to secure more than small fragments.

The specimens illustrated in plate 1, figures 2 to 7, were collected from Rock Crossing in the Criner Hills. The one illustrated in figures 1, 1*a* was collected $1^1/_2$ miles west of Nebo Store southwest of Sulphur. This largest fragment measures 21 mm. in length, and the entire length to which the rhabdosomes grew is not known. The width at the sicular end is 1 mm. and this increases in the distance of 6 thecae to 2 mm. which is about the maximum width. A narrow line-like median depression occurs on the obverse side. The thecae number 10 to 14 in 10 mm. and overlap about one-half their length. Angle of slope is about 25°. The margins are deeply notched. Two delicate lateral spines occur on the proximal end. The sicula, a little over 1 mm. long, may be seen in figure 6*a*.

This new species occurs most abundantly at Rock Crossing in the Criner Hills, 8 miles southwest of Ardmore. The specimen illustrated in figure 1 was secured from $1^1/_2$ miles west of Nebo Store, 11 miles southwest of Sulphur; a specimen was found on Cool Creek, 19 miles northeast of Ardmore; and some fragments were found on the West Branch of Sycamore Creek at the same horizon, though no certain identification was made.

This new species of graptolite is very important for local correlation, as it, with other definite horizon markers, relates the Bromide formation in the upper part of the Simpson group in the different regions in a very clear way, and helps to demonstrate in a significant manner the unity and individuality of the Bromide formation as developed in the Criner Hills and Arbuckle Mountains of Oklahoma.

* Acknowledgment is made to the NATIONAL RESEARCH COUNCIL for grants to meet part of the expenses of preparation of materials for this and four other papers, to Dr. Rudolf Ruedemann for assistance with the drawings, and to Rex McGehee and Ross Maxwell for field assistance.

[1] Decker, C. E., Oklahoma Geol. Surv. *Bull.*, **55**, 16 (1931).
[2] Decker, C. E., *A. A. P. G.*, **17**, 1405–1435 (1933).
[3] Ruedemann, R., and Decker, C. E., *Jour. Paleon.*, **8**, 303–327 (1934).
[4] Ulrich, E. O., *G. S. A.*, **22**, 663 (1911).
[5] Elles, G. L., and Wood, E. M. R., *British Grapht.*, **49** (1901–1918).
[6] Hall, Jas., *Can. Org. Rem.*, **2**, 73, 74 (1865).
[7] Ruedemann, R., *Grapt. of N. Y.*, **1**, 690 (1904).
[8] Gurley, R. R., *Jour. Geol.*, **4**, 295 (1896).
[9] Elles, G. L., and Wood, E. M. R., Op. cit., **43, 44** (1901–1918).
[10] Ruedemann, R., Op. cit., 691 and op. cit., 490 (1904).
[11] Elles, G. L., and Wood, E. M. R., Op. cit., **48, 49** (1901–1918).
[12] Elles, G. L., and Wood, E. M. R., Op. cit., **221** (1901–1918).

THE LOCATION OF A GENE FOR DISEASE RESISTANCE IN MAIZE

BY VIRGINIA H. RHOADES*

CORNELL UNIVERSITY

Communicated March 18, 1935

The inheritance of resistance to physiologic forms 1 and 3 of rust, *Puccinia Sorghi* Schw., of corn has been shown by Mains[1] to be dependent in each case on a single Mendelian factor. The inheritance of resistance in relation to various other Mendelian factors was studied but no indication of linkage was obtained.

The study reported here is concerned with the determination of the chromosomal location of the factor for resistance to physiologic form 3 of *Puccinia Sorghi*. The location of the factor was determined cytologically by means of x-ray induced deficiencies, following the method used by McClintock,[2] and then checked genetically by means of trisomic ratios.

Pollen from a plant homozygous for the dominant factor for resistance was treated with x-rays (dose 1000 or 2000 *r* units) and then used to pollinate susceptible plants. The seed obtained from these pollinations was grown

in the greenhouse during the winter months and the seedlings inoculated with the rust. These F_1 seedlings normally would be heterozygous for the dominant factor for resistance. Any susceptible seedling would probably be the result either of a contamination or of a pollination by a pollen grain which had lost because of the x-ray treatment the factor for resistance. Resistant and susceptible plants were easily distinguishable. In all the cultures used in this study classification was made when the susceptible plants were showing spores. No spores developed on the resistant plants, but the leaves showed white necrotic flecks or spots where the fungus had penetrated and the leaf tissue had died. Since both susceptible and resistant plants showed definite but different reactions to the fungus it was possible to determine when a plant had escaped inoculation. A number of susceptible plants were obtained by the method described. These were grown and examined cytologically at the mid-prophase of meiosis. A few either had no visible alteration of the chromosomes or were the result of contamination. Some five or six had several chromosomes injured and showed at meiosis very complex configurations. No attempt was made to study these in detail. Four of the susceptible plants, however, showed a deficiency in the short arm of the tenth and shortest chromosome of the haploid set. One of these plants had in addition a deficiency in one of the longer chromosomes. The four deficiencies in chromosome 10 were all apparently terminal. However, since non-homologous association of chromosomes is known to occur in maize, it is possible that the deficiencies were internal with non-homologous association occurring between the normal and the deficient chromosomes. Camera lucida drawings of

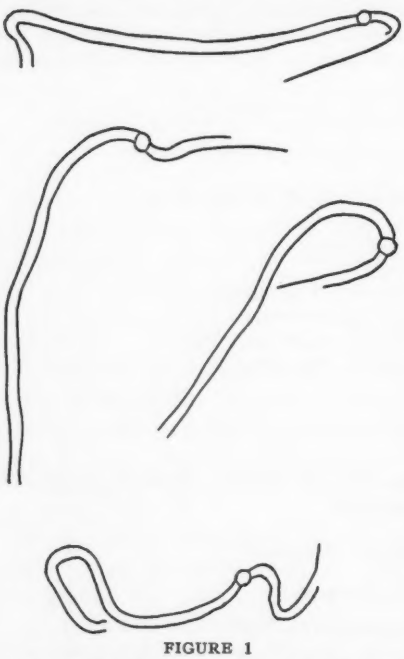

FIGURE 1

Diagrammatic representations based on camera lucida drawings at mid-prophase of each of the four deficiencies in the short arm of chromosome 10. The spindle fibre attachment region is represented by a small bulge. Magnification approximately 1900 times.

the four deficiencies at mid-prophase are shown in figure 1. These deficiencies suggested that the factor for resistance to physiologic form 3 of *Puccinia Sorghi* was located in the short arm of chromosome 10.

The conclusion derived from the cytological study was verified genetically by means of trisomic ratios. In maize eight of the ten primary trisomic types have been obtained. Cultures of each of these were tested with the rust and all proved to be susceptible. Trisomic plants from each of the eight cultures were then used as the female parent in crosses with resistant plants. Pollen from F_1 $2n + 1$ plants was then used in pollinating susceptible plants. The seedlings from these latter crosses were grown in the greenhouse, inoculated with the rust and classified as resistant or susceptible. The ratio of resistant to susceptible seedlings should be 1:1 in those crosses in which the chromosome present in trisomic condition was not the chromosome which carries the gene for resistance. But if in any of the crosses the trisomic chromosome was the one which carries the factor for resistance the ratio should be 1 resistant:2 susceptible plants ($n + 1$ pollen grains cannot successfully compete in achieving fertilization with haploid (n) grains, and among the haploid grains from a trisomic plant carrying one dominant and two recessive genes there are two pollen grains with the susceptible factor to one with the resistant factor). The results of these crosses are shown in table 1. Seven of the eight trisomic crosses gave 1:1 ratios while the eighth, which involved chromosome 10, gave a ratio of 1 resistant:2 susceptible plants, indicating that this chromosome carries the factor for resistance to the form of rust being studied.

TABLE 1

RESULTS OF CROSSING SUSCEPTIBLE PLANTS BY TRISOMIC PLANTS CARRYING A SINGLE DOMINANT GENE FOR RESISTANCE. TRISOMIC PLANTS WERE USED AS THE MALE PARENT

TRISOMIC CHROMOSOME	RESISTANT PLANTS	SUSCEPTIBLE PLANTS
Chromosome 2	239	217
Chromosome 3	246	271
Chromosome 5	279	286
Chromosome 6	375	319
Chromosome 7	191	183
Chromosome 8	198	225
Chromosome 9	279	284
Chromosome 10	418	763

D/P.E. values were calculated for all the results and only in one case, with the exception of chromosome 10 where a 1:2 ratio was obtained, could the deviation from a 1:1 ratio be considered significant, and the deviation found for chromosome 6 was not in the direction to indicate linkage of the factor for resistance and that chromosome.

In order to test the two chromosomes which were not available in trisomic condition, crosses were made with factors known to be located in those chromosomes. The tests for chromosome 1 are not yet complete. The test for chromosome 4 was as follows: *su* (sugary) gl_3 (glossy$_3$) susceptible plants were crossed to *Su* Gl_3 resistant plants and the F_1 plants backcrossed to *su* gl_3 susceptible individuals. The results obtained are shown in table 2. Chi square values were calculated for the results and the deviations were found not to be significant. These results indicate that the factor for resistance is independent in its inheritance of the factors *su* and gl_3. This agrees with the results of Mains who tested *su* and *Tu*, another gene in chromosome 4, and found them to be independent of the factor for resistance to physiologic form 3 of the rust.

TABLE 2

BACKCROSS DATA FROM LINKAGE TESTS OF THE RESISTANT FACTOR AND GENES IN CHROMOSOME 4

CULTURE	RES. Su	RES. su	SUSC. Su	SUSC. su
25-11 × 23-16	31	22	24	22
23-12 × 25-4	37	30	26	31
25-1 × 23-4	46	52	47	50
Total	114	104	97	103
	RES. Gl_3	RES. gl_3	SUSC. Gl_3	SUSC. gl_3
25-1 × 23-4	53	45	45	52

Studies of the linkage relations between this factor for rust resistance and other genes in the tenth linkage group are in progress to determine the exact position of this gene in the genetic map.

Summary.—Cytological studies of x-ray induced deficiencies and genetical studies of trisomic ratios indicate that the factor for resistance to physiologic form 3 of *Puccinia Sorghi* is located in the short arm of the tenth and shortest chromosome of the haploid maize complement.

* I wish to thank Professor E. B. Mains of the University of Michigan for his generosity in supplying the form of rust studied and resistant and susceptible strains of maize.

[1] Mains, E. B., *Jour. Agr. Res.*, **43**, 419–430 (1931).

[2] McClintock, Barbara, *Mo. Agr. Exp. Sta. Bull.*, 163 (1931).

LINKAGE RELATIONS OF ZAVADSKAIA SHAKER IN THE HOUSE MOUSE (*MUS MUSCULUS*)

BY FRANK H. CLARK

LABORATORY OF VERTEBRATE GENETICS, UNIVERSITY OF MICHIGAN

Communicated April 13, 1935

A recessive gene mutation of the house mouse which appeared among the descendants of individuals subjected to x-ray treatment, has been reported by Dobrovolskaïa-Zavadskaïa (1928). Although described as a new waltzing mutation, animals with the anomaly seem to be indistinguishable in behavior from those possessing the well-known shaker character reported by Lord and Gates (1929). According to Gates (1934), this new mutation (sh_2) is genetically different from ordinary shaker although it resembles it in every respect.

A preliminary study of the linkage relations of this new shaker gene were begun and reported by Gates (1934). When Gates found that I had independently started a similar study, he discontinued his tests and kindly gave me permission to complete the study. In this way duplication of effort has been avoided and our results with one or two exceptions are complementary. This study was begun at the Bussey Institution, Harvard University, and completed in the Laboratory of Vertebrate Genetics, University of Michigan.

Gates found that this new shaker mutation is independent of, and segregates normally from the following characters: agouti (*A*), pink-eye (*p*), dilution (*d*), short-ear (*se*), black-eyed white (*W*) and albinism (*c*). With the exception of the agouti and black-eyed-white cross, his tests were made by using the F_2 generation. For the sake of brevity, I shall not include his results in this paper, but can say unreservedly that his conclusions are supported by ample numbers of F_2 and backcross animals.

It has already been shown (Clark, 1934a, 1934b) that this shaker (sh_2) is not linked with albinism, brachyury or flexed-tail anemia.

In the present study, the new shaker was tested for linkage with the following characters: waltzing (*v*), Robert's pink-eye (p_2), chocolate (*b*), leaden (*l*), dwarf (*dw*), hairless (*hr*), rodless (*r*), hydrocephalus$_1$, and naked (*Nk*). In the test with naked a backcross was used (table 2) but with the other characters F_2 populations were utilized (table 1).

Before discussing the results of the present experiments, it would be well to present evidence to show that the new shaker is different genetically from ordinary shaker. Gates crossed animals homozygous for the new shaker (sh_2) with mice homozygous for ordinary shaker (sh_1), and obtained normal F_1 young in every case. Dunn (unpublished data) informs me that he has likewise made the same cross and obtained the same results.

Further evidence of the genetic dissimilarity of these two shaker characters is had from the negative results secured in linkage tests of the new shaker with pink-eye and albinism. It is well established that, in the house mouse, the genes for ordinary shaker (sh_1), albinism and pink-eye are located in the same chromosome (Haldane, *et al.*, 1915; Lord and

TABLE 1

F_2 DATA FROM LINKAGE TESTS BETWEEN ZAVADSKAIA SHAKER (sh_2) AND OTHER MUTANT CHARACTERS OF THE HOUSE MOUSE

The expected 9:3:3:1 ratio and the deviations divided by their respective probable errors are given for each cross. *X* indicates the dominant allelomorph of the character tested, *x* its recessive allelomorph.

		X Sh_2	*x* Sh_2	*X* sh_2	*x* sh_2	TOTAL NO.
Pink-eye$_2$ (Roberts)	Observed	55	13	16	6	90
	Expected	50.6	16.9	16.9	5.6	
	Dev./P.E.	1.39	1.56	0.36	0.26	
Chocolate	Observed	60	24	24	10	118
	Expected	66.4	22.1	22.1	7.4	
	Dev./P.E.	1.76	0.66	0.66	1.47	
Leaden	Observed	53	27	17	10	107
	Expected	60.2	20.1	20.1	6.7	
	Dev./P.E.	2.08	2.54	1.14	1.95	
Dwarf	Observed	49	12	23	5	89
	Expected	50.0	16.7	16.7	5.6	
	Dev./P.E.	0.32	1.90	2.54	0.40	
Hairless	Observed	84	22	16	7	129
	Expected	72.6	24.2	24.2	8.1	
	Dev./P.E.	3.00	0.74	2.74	0.60	
Rodless	Observed	39	17	10	3	69
	Expected	38.8	13.0	13.0	4.3	
	Dev./P.E.	0.07	1.87	1.40	1.00	
Hydrocephalus (hy_1)	Observed	33	18	15	0	66
	Expected	37.1	12.4	12.4	4.1	
	Dev./P.E.	1.51	2.61	1.22	3.08	

Gates, 1929). If the new shaker (sh_2) were identical with ordinary shaker (sh_1) we should expect that it, too, would exhibit linkage with albinism and pink-eye. Gates's results, however, show conclusively that the new shaker is not linked with either of these characters. My own test with chinchilla, an albino allelomorph, reported in a later paper (Clark, 1934b)

agrees with Gates's data. There is no reasonable doubt, then, that these two phenotypically similar shaker anomalies owe their expression to different genes located on different chromosomes.

Since shaker mice sometimes whirl in circles like waltzers it seemed important to cross mice having these two anomalies. In every case a normal F_1 was obtained, showing that the two characters have a different genetic basis. Gates and also Dunn (unpublished data) later informed me that they had made the same test with identical results. It is difficult in F_2 to separate waltzers, shakers and waltzer-shakers with certainty, so these three classes were combined. If there is no linkage between shaker (sh_2) and waltzing we should expect in F_2 a ratio of 9 normal to 7 abnormal. If these two characters were linked, however, in a repulsion cross of this type we should expect a deficiency of normal mice and an excess of shakers and waltzers. Close linkage should give nearly equal numbers of normals and abnormals in F_2. A ratio was obtained of 94 normals to 54 showing one or both anomalies. This deviates from the expected 9:7 ratio by 10.8 which is 2.64 times its probable error. But it is opposite in direction to a deviation which linkage would produce, since there is a deficiency instead of an excess of abnormals. Such a deviation as that observed could easily be due to chance and would be expected once in 13 trials. It is quite possibly due to differential mortality. This linkage test indicates clearly that the genes for shaker (sh_2) and waltzing lie in different chromosomes.

The F_2 ratios obtained from linkage tests with Robert's pink-eye, chocolate, leaden, dwarf, hairless and rodless give negative results (table 1) in so far as possible linkage is concerned. In every case, the deviation of the double recessive class from the expected number is less than two

TABLE 2

BACKCROSS DATA FROM A LINKAGE TEST BETWEEN ZAVADSKAIA SHAKER (sh_2) AND NAKED (REPULSION CROSS)

	Sh_2		sh_2		
	Nk	*nk*	*Nk*	*nk*	TOTAL NO.
Observed	26	28	30	25	109
Expected	27.25	27.25	27.25	27.25	
Dev./P.E.	0.41	0.25	0.93	0.74	

times its probable error and, therefore, not significant. In the test with hairless a slight excess of normals and shaker (non-hairless) mice was secured, the deviations being respectively 3.00 and 2.74 times their own probable errors. Such deviations would occasionally be expected from chance variation and, in this case, can be ignored because they are in the wrong direction to indicate linkage. Moreover, the double-recessive class fits the expected number very closely, which in itself indicates that shaker and hairless are independent characters. The results of the back-

cross involving shaker and naked are shown in table 2. The observed data fit the expected 1:1:1:1 ratio very closely and it is evident that shaker is not linked with naked.

Hydrocephalus, another mutation (Clark, 1932), has not as yet been completely tested for linkage with the other mutant characters in the house mouse. It has been shown (Clark 1935) that this type of hydrocephalus is genetically distinct from another (which we may call hydrocephalus$_2$) reported by Zimmermann (1933) in Germany. The new shaker is being tested for possible linkage with the original type of hydrocephalus. Preliminary results from a repulsion cross shown in table 1 strongly indicate close linkage between the new shaker and hydrocephalus. As yet, no double recessive hydrocephalus-shaker animals have been ob-

TABLE 3

CHROMOSOMES OF THE HOUSE MOUSE AT PRESENT TAGGED WITH IDENTIFYING MUTANT GENES, WITH APPROXIMATE MAP DISTANCES OF SUCH GENES

1.	Shaker (sh_1)	0	8.	Waltzing
	Albino series	3		
	Pink-eye$_1$	14		
2.	Agouti series	0	9.	Chocolate
	Pink-eye$_2$ (Roberts)	12		
3.	Hairless	0	10.	Leaden
	Piebald	10		
4.	Rodless	0	11.	Dwarf
	Silver	10		
5.	Flexed-tail		12.	Naked
	Anemia			
	Bellyspot			
6.	Dilution	0	13.	Brachyury
	Short-ear	0.01		
7.	Black-eyed-white		14.	Shaker (sh_2) hydrocephalus ?
			15–20—Untagged	

served where 4.1 would be expected. This deviation is 3.08 times its probable error, and possibly significant. Moreover, the F_2 ratio approximates very closely the 2:1:1:0 ratio which is indicative of close linkage. Unfortunately, a backcross cannot be used because of the partially lethal action of the hydrocephalus gene. A satisfactory linkage test of these characters will require a large F_2 population and a determination of the genotypes of many of the F_2 shakers. If linkage does not exist, two-thirds of the F_2 shaker animals should be heterozygous for hydrocephalus and carry one crossover gamete. The results of these tests will be reported later in a paper on the linkage relations of hydrocephalus.

In order to summarize the present status of linkage tests in the house mouse and the scope of the work described in the present experiment, table 3 is presented. In this table, each of the 20 chromosomes of the

mouse is listed by number (1–20), and the characters or groups of characters which serve as chromosome markers are listed opposite these numbers, with their respective map distances. Untagged chromosomes are listed by number only.

Before the present study was made, 13 of the 20 chromosomes of the mouse were tagged with one or more genes. Gates has tested the new shaker with one or more characters on chromosomes 1, 2, 6 and 7 with negative results. In the present and previous linkage studies, I have tested shaker with mutant genes located on chromosomes numbers 1–13 and have shown that the gene for the new shaker apparently does not lie in any one of these chromosomes. It logically follows that since the gene for shaker does not lie in one of the 13 tagged chromosomes, it must be located on a fourteenth chromosome, one of the 7 untagged ones. If shaker (sh_2) proves to be linked with hydrocephalus, as the preliminary data indicate, the gene for hydrocephalus would also lie on this same fourteenth chromosome. The number of independent genes or groups of genes in the house mouse has thus been increased from 13 to 14 by the present investigation while 6 chromosomes remain to be tagged.

Conclusions.—1. The Zavadskaïa shaker character in the house mouse (*Mus musculus*) is genetically distinct from ordinary shaker (sh_1) and from waltzing, and is not linked with either of these characters.

2. The new shaker (sh_2) has been tested for linkage with 16 other characters which serve as markers for 13 of the 20 chromosomes of the mouse. Since no linkage was observed with any of these characters, it follows that the gene for the new shaker must be located in a fourteenth chromosome. Preliminary linkage tests indicate that the gene for hydrocephalus (hy_1) may lie in this same chromosome.

Clark, F. H., "Hydrocephalus, a Hereditary Character in the House Mouse," *Proc. Nat. Acad. Sci.*, **18,** 654–656 (1932).

Clark, F. H., "Linkage Studies of Brachyury (Short-Tail) in the House Mouse," *Ibid.*, **20,** 276–279 (1934a).

Clark, F. H., "The Inheritance and Linkage Relations of a New Recessive Spotting in the House Mouse," *Genetics*, **19,** 365–393 (1934b).

Clark, F. H., "Two Hereditary Types of Hydrocephalus in the House Mouse (*Mus musculus*)," *Proc. Nat. Acad. Sci.*, **21,** 150–152 (1935).

Dobrovolskaïa-Zavadskaïa, N., "L'irradiation des testicules et l'hérédité chez la souris," *Arch. Biologie*, **38,** 457–501 (1928).

Gates, W. H., "Linkage Tests of the New Shaker Mutation with Other Factors in the House Mouse (*Mus musculus*)," (Abstract), *Am. Nat.*, **68,** 173 (1934).

Haldane, J. B. S., *et al.*, "Reduplication in Mice," *Jour. Genet.*, **5** (1915).

Lord, E., and Gates, W. H., "Shaker, a New Mutation of the House Mouse," *Amer. Nat.*, **63,** 435–442 (1929).

Zimmermann, Klaus, "Eine neue Mutation der Hausmaus: 'Hydrocephalus,'" *Zeit. ind. Abstam. Vererb.*, **64,** 176–177 (1933).

GENERATION OF ANY N-VALUED LOGIC BY ONE BINARY OPERATION[1]

BY DONALD L. WEBB

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated April 1, 1935

A two-valued logic was shown in 1913 by Sheffer[2] to be obtainable by the iteration of a single binary operation. It was proved in 1925 by Zylinski[3] that Sheffer's function and its "dual," also introduced by Sheffer, are the only binary operations such that the iteration of either one will generate the two-valued logic of functions of two propositions. Zylinski's proof was by means of a truth table of 4 columns and 16 rows, corresponding to the two possible values (a or b) which an arbitrary function $\Phi(x, y)$ can assume in a two-valued system. While theoretically applicable to an n-valued system, the method of direct inspection of the truth table is impracticable. Following another method we prove that: *Any n-valued logic, where $n \geqq 2$, can be generated by the iteration of one binary operation.*

Designate the n truth values which an "elementary proposition" may take in an n-valued logic by the marks $a_0, a_1, \ldots, a_{n-1}$. For convenience, drop the a and retain only the subscript, so that our marks are now $0, 1, \ldots, n-1$. It is to be observed that these numbers denote merely n distinct marks without any arithemetical significance. Let p and q be any elementary propositions. Construct a truth table for two elementary propositions, p, q, of two columns and n^2 rows with the n marks, $0, 1, \ldots, n-1$ by assigning in the ith row of the table to p the value $[i-1-(i-1)']/n$, and to q the value $(i-1)'$, where $j' \equiv j \bmod n$, $j' \geqq 0$ and $i = 1, 2, \ldots, n$. Denote the statement, *p has the value i*, by $p = i$; let $p\beta q$ denote any function of p, q whose values are in the range $0, 1, \ldots, n-1$, when $p = i, q = j$ and i, j are in the same range; let $i\beta j = k$ denote that if $p = i$ and $q = j$ then $p\beta q = k$, where k is in the range.

Define the stroke, "$|$," function, $p|q$, by

$$i|j = 0 \text{ if } i \neq j; \quad i|i = (i+1)' \quad (i, j = 0, 1, \ldots, n-1).$$

From this binary operation we shall generate all functions of two variables in the n-valued logic. The proof will consist in exhibiting the particular general column of the p, q truth table in which the n^2 marks t_s ($s = 1, 2, \ldots, n^2$) are arbitrary elements of the set $0, 1, \ldots, n-1$ as a function constructed on p, q by means the stroke $|$.

As a notational definition we write

$$p^0 \equiv p, \quad p^i \equiv p^{i-1}|p^{i-1} \quad (i = 1, \ldots, n-1),$$

in which the exponents are superscripts. Define $R_{i,j}(p, q)$ to be a function of p and q such that in row j of the truth table, $R_{i,j}(p, q) = i$ and in every

other row $R_{i,j}(p, q) = 0$ $(i = 0, 1, \ldots, n-1;\ j = 1, 2, \ldots, n^2)$. For brevity we shall write $R_{i,j}$ for $R_{i,j}(p, q)$. Similarly, any function $F(p, q)$ of p, q will be written F.

From the definition of $p|q$ it is evident that p^1 can be obtained by operating with the cyclic substitution $S \equiv (0,1\ldots, n-1)$ on the values of p^0. Similarly, p^i can be obtained by operating on the values of p^0 with the ith power, S^i, of S. Hence $R_{1,j}$ and $(p^a|q^b)^{n-1}|(p^0|p^1)$ are equivalent for the values of a, b indicated next, which we shall write

$$R_{1,j} \cdot\equiv\cdot (p^a|q^b)^{n-1}|(p^0|p^1),\ a = \{n-[i-1-(i-1)']/n\}',\ b=[n-(i-1)']'.$$

By means of $R_{1,j}$ we obtain $R_{i,j}$ in terms of p, q and $p|q$. For,

$$R_{0,j} \cdot\equiv\cdot p^0|p^1,\ R_{i,j} \cdot\equiv\cdot (R_{1,j})^{i-2}|(R_{0,j})^{i-1}\ (i = 2, 3, \ldots, n-1).$$

Define $p\alpha_1 q \equiv p^0|q^{n-1}$, $p\alpha_2 q \equiv R_{1,1}|R_{0,1}$, $p\alpha_i q \equiv N_{i-2}\alpha_1 M^1_{i-2}$ $(i = 3, 4, \ldots, n)$. $M_0 \equiv R_{0,1}$, $M_h \equiv M_{h-1}\alpha_{h+1}(R_{h,h+2}\alpha_{h+1}R_{h,(h+1)n+1})$, $N_h \equiv M_h\alpha_{h+1}R_{1,1}$.

Then we see that $p\alpha_3 q$ has the property $0\alpha_3 i = i;\ i\alpha_3 0 = i$ $(i = 0, 1, 2)$. Hence, by an easy mathematical induction, we prove that if

$$p\alpha_n q \equiv N_{n-2}\alpha_1 M^1_{n-2}, \text{ then } 0\alpha_n i = i;\ i\alpha_n 0 = i\ (i = 0, 1, \ldots, n-1).$$

Let $T_i(p, q)$ be any function of p, q and define

$$T_1\alpha_n T_2\alpha_n T_3 \equiv (T_1\alpha_n T_2)\alpha_n T_3$$
$$T_1\alpha_n T_2\alpha_n T_3\alpha_n T_4 \equiv ((T_1\alpha_n T_2)\alpha_n T_3)\alpha_n T_4$$
$$\ldots \qquad\qquad \ldots$$
$$T_1\alpha_n T_2\alpha_n T_3\alpha_n \ldots \alpha_n T_j \equiv (\ldots((T_1\alpha_n T_2)\alpha_n T_3)\alpha_n \ldots \alpha_n T_j.$$

From the properties of $p\alpha_n q$ it follows that

$$0\alpha_n 0\alpha_n \ldots \alpha_n 0\alpha_n i\alpha_n 0\alpha_n \ldots \alpha_n 0 = i\ (i = 0, 1, \ldots, n-1).$$

Hence we can construct any function $F(p, q)$ of p, q having the values t_i in row i of the truth table, where t_i is any one of the marks $0, 1, \ldots, n-1$, as follows:

$$F(p, q) \cdot\equiv\cdot R_{t_1,1}\alpha_n R_{t_2,2}\alpha_n \ldots \alpha_n R_{t_{n^2},n^2}$$

This follows immediately from $p\alpha_n q$ by inspection of the rows in the following table,

$$t_1\alpha_n 0\alpha_n 0\alpha_n \ldots \alpha_n 0\alpha_n \ldots \alpha_n 0 = t_1$$
$$0\alpha_n t_2\alpha_n 0\alpha_n \ldots \alpha_n 0\alpha_n \ldots \alpha_n 0 = t_2$$
$$\ldots \qquad\qquad \ldots$$
$$0\alpha_n 0\alpha_n 0\alpha_n \ldots \alpha_n t_i\alpha_n \ldots \alpha_n 0 = t_i$$
$$\ldots \qquad\qquad \ldots$$
$$0\alpha_n 0\alpha_n 0\alpha_n \ldots \alpha_n 0\alpha_n \ldots \alpha_n t_{n^2} = t_{n^2}$$

Now $F(p, q)$ is any function defined by the truth table of the n-valued

logic. Hence, if $n \geqq 2$, *any* function of two propositions p, q can be constructed from p, q and $p|q$ in the manner just indicated. To extend this result to a function of m propositions, $p_1, p_2, \ldots, p_m$, we make the additional definitions and take the steps indicated below.

Define $i_{r,n} \equiv i \bmod n^r$ where i is any positive integer and $i_{r,n} \geqq 0$. Now we shall construct a truth table on $p_1, \ldots, p_k$ by assigning to p_s $(s = 1, 2, \ldots, k)$ the value

$$p_s = \{[(j-1) - (j-1)_{k-s,n}]/n^{k-s}\}_{1,n}$$

in the jth row $(j = 1, 2, \ldots, n^k)$. Let $R^k_{i,j} \equiv R_{i,j}(p_1, \ldots, p_k)$ be defined on a truth table of $p_1, \ldots, p_k$ as $R^k_{i,j} = i$ in row j, and $R^k_{i,j} = 0$ elsewhere. Let $S^k_j \equiv (R^k_{n-1,j})^1$ $(j = 1, 2, \ldots, n^k)$.

It is evident that a function $F(p_1, \ldots, p_k)$ will exist in a truth table on $p_1, \ldots, p_{k+t}$ and will be the function obtained from the $p_1, \ldots, p_k$ truth table by replacing each row by n^t consecutive identical rows. This, then, increases the number of rows from n^k to n^{k+t}.

Hence, we can prove

$$R^k_{i,j} \cdot\equiv\cdot R_{i,h}(S^{k-1}_l, p_k) \text{ where } h = (j-1)_{1,n} + 1,\ l = [j-1-(j-1)_{1,n}]/n + 1 \ (j = 1, 2, \ldots, n^k;\ k = 3, 4, \ldots, m).$$

Having determined $R^m_{i,j}$, we proceed to construct any $F(p_1, \ldots, p_m)$ in terms of $p_1, \ldots, p_m$ and the stroke, "|", by using the same procedure as for $m = 2$. For if t_i is the value of $F(p_1, \ldots, p_m)$ in row i, then it follows that

$$F(p_1, \ldots, p_m) \cdot\equiv\cdot R^m_{t_1,1}\alpha_n R^m_{t_2,2}\alpha_n \ldots \alpha_n R^m_{t_n,n}.$$

Thus any function on any number of propositions $p_1, \ldots, p_m$ can be constructed by means of a single binary operation, the stroke, "|".

[1] The author is under obligations to Dr. E. T. Bell for his suggestions, aiding both in the solution of this problem and its presentation.

[2] Sheffer, H. M., *Trans. Amer. Math. Soc.*, **14**, 481–488 (1913).

[3] Zylinski, E., *Fund. Math.*, **7**, 203–209 (1925).

GENERALIZED DERIVATIVES AND APPROXIMATION

BY W. E. SEWELL

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

Communicated April 8, 1935

The purpose of this paper is to state briefly results on approximation by polynomials to functions analytic in a region and continuous in the closed region,[1] where the continuity properties are specified by the order of the generalized derivative.[2]

1. *Introduction.*—Let C be a rectifiable Jordan arc or curve in the z-plane, let k be an arbitrary but fixed point on C, and let $f(z)$ be defined on C. The generalized derivative of $f(z)$ is defined as follows:[3]

$$\begin{aligned} D_z^0 f(z) &= f(z), \\ D_z^\alpha f(z) &= \frac{1}{\Gamma(-\alpha)} \int_k^z (z-x)^{-\alpha-1} f(x)dx, \ \alpha < 0, \\ D_z^\alpha f(z) &= \frac{d^p}{dz^p} D_z^{\alpha-p} f(z), \ 0 \leqq p-1 \leqq \alpha < p, \end{aligned} \tag{1}$$

where p is a positive integer and the path of integration is along C. For functions analytic in a region containing C we may change the path of integration by virtue of the following:

THEOREM 1.1. *Let $f(z)$ be regular on and within a contour γ which passes through k and contains in its interior that portion of C between k and z. Then for $\alpha > 0$,*

$$D_z^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_\gamma \frac{f(t)dt}{(t-z)^{\alpha+1}}. \tag{2}$$

If $\int_{z_1}^{z_2} |z_2 - z|^{-\alpha} |dz|$, $0 < \alpha < 1$, z_1, z_2 any two points on C, the path of integration being along C, converges uniformly, we shall say that C is of type W. If C is of type W and every point of C is infinitely accessible we have the additive property for generalized derivatives:

$$D_z^{-\alpha} D_z^\alpha f(z) = f(z). \tag{3}$$

We shall also make use of the following:

THEOREM 1.2. *If $f(z)$ has a bounded generalized derivative of order α it has a bounded generalized derivative of any order $\alpha' < \alpha$.*

2. *Analytic Curves.*—Let the region R be finite and let its boundary C be an analytic Jordan curve. Under these conditions we have:

THEOREM 2.0. *If $f(z)$ has a bounded generalized derivative of order α on C, then it has a bounded generalized derivative of the same order after conformal transformation.*

By an application of this fact and the expression (2) the following generalization of Bernstein's theorem[4] can be established:

THEOREM 2.1. *If $P_n(z)$ is a polynomial of degree n in z and $|P_n(z)| < M$ in a region bounded by an analytic Jordan curve C, then $|P_n^\alpha(z)| < Mk(\alpha, C)n^\alpha$, $\alpha > 0$, on the curve C.*

Theorems proved by Kellogg[5] and Seidel[6] serve to *extend this result to curves whose tangents turn with sufficient continuity.*

By proving that the uniform convergence of $P_n(z)$ to $f(z)$ and the uniform convergence of $P_n^\alpha(z)$ imply the existence of $f^\alpha(z)$ we are able to show that:

THEOREM 2.2. *If for every n there exists a polynomial $P_n(z)$ of degree n such that $|f(z) - P_n(z)| < M/n^\alpha$, $\alpha > 0$, on and within the analytic Jordan curve C, then $f(z)$ has derivatives of all orders $\alpha' < \alpha$ on C.*

This, in conjunction with previous results of the author,[7] leads to the following:

THEOREM 2.3. *If a function $f(z)$ is continuous in a closed region bounded by an analytic Jordan curve C, is analytic in the interior, and satisfies a Hölder condition of order α, $0 < \alpha \leqq 1$, on C, then $f(z)$ has a bounded generalized derivative of any order $\alpha' < \alpha$ on C.*

By making use of (3) and theorem 2.0 we obtain:

THEOREM 2.4. *If $f(z)$ has a bounded generalized derivative of order α, $0 < \alpha \leqq 1$, on C, an analytic Jordan curve, then it satisfies a Hölder condition of order α on C.*

Theorems 2.3 and 2.4 extend to the ordinary derivatives of the function and values of α greater than unity.

3. *Curves with Corners.*—Let the region R be finite and bounded by a Jordan curve C composed of a finite number of analytic Jordan arcs. Let the corners of C be denoted by $z_1, z_2, \ldots, z_r$, let the exterior opening at the corner z_j be $\mu_j\pi$, $0 < \mu_j < 2$, and let $2 > \mu_1 \geqq \mu_2 \geqq \ldots \geqq \mu_r > 0$. Then we define t as follows:

$$t = \mu_1 \quad \text{if} \quad \mu_1 \geqq 1;$$
$$t = 1 \quad \text{if} \quad \mu_1 < 1.$$

We can show that C_ρ (Green's level curve for the region exterior to C) lies outside of a circle about any point of C of radius $N(\rho - 1)^t$ (an inequality in the other direction can also be established). This enables us by (2) to prove an extension of theorem 2.1.

THEOREM 3.1. *If $|P_n(z)| < M$ on C, then $|P_n^\alpha(z)| < Mk(\alpha, C)n^{\alpha t}$, $\alpha > 0$, on C.*

If we replace α in theorem 2.2 by αt we have the corresponding result for this case, and the proof goes through without change. The method of proof *in theorem 2.3* will not hold here, but *a proof of Hardy and Littlewood[8] for real variables can be extended to this case. By consideration of the analyticity of the arcs of C and breaking the boundary up into its component*

parts we are enabled to extend theorem 2.4. Generalizations as to smooth arcs go through just as in the preceding paragraph.

4. *Rectifiable Curves.*—Here the boundary is assumed to be a rectifiable Jordan curve C, however, further restrictions must be put on the region in order to apply the present methods. Along a Jordan curve or arc γ consider the integral:

$$I = \int_{z_1}^{z_3} |z - z_3|^{-\beta} |dz|, \beta > 0,$$

where the *point* z_2 *lies between* z_1 *and* z_3. *If there exists a constant* M *such that for* z_2 *arbitrarily near* z_3 *we have*

$$I < M |z_2 - z_3|^{-\beta+1},$$

then we shall say that z_3 *is a linear point of* γ. *The constant* M *is a function of* z_3 *and if this function is uniformly bounded for all points of* γ *we shall say that* γ *is a curve or arc of type* W'.

As in the previous case we need to locate the *Green's level curves* for the exterior region and here the *order of the distance away from the boundary is* $(\rho - 1)^2$: i.e., $t = 2$. The inequality in one direction can be established by a transformation of the unit circle into itself and an application of the "Verzerrungsatz"; this evaluation holds for an arbitrary simply connected region and is the one which is needed in connection with the proofs of the theorems here. The inequality in the other direction can be established for a boundary point which can be joined to the point at ∞ by a straight line; the proof is a modification of Lindelöf's method.[9]

Now if the region is bounded by a curve of type W' *we get a result corresponding to theorem* 3.1 *with* $t = 2$. Such a boundary, however, is not necessary due to (2); if the point k can be joined to the boundary point z by a curve of type W' lying inside or on the boundary of the region we have the above evaluation at the point z, and if the function M is uniformly limited for all boundary points and suitable corresponding arcs we have

$$|P_n^\alpha(z)| < Mk(\alpha, C)n^{2\alpha}, \text{ on } C.$$

In fact the evaluation holds if the end-point of the arc joining k *to* z *is linear and the quantity* M *is uniformly limited for all boundary points when* $|z_2 - z| < \epsilon$, z *being the end-point,* z_2 *a point on the arc, and* ϵ *a fixed positive quantity independent of* z.

A generalization of theorem 2.2 *can be proved for the same regions* as described in the above paragraph, *the exponent of* n *being* 2α.

If the boundary of the region is a curve of type W *we get a generalization of theorem* 2.3; *the proof is the same.*

5. *Closed Sets.*—By means of the above results we can prove the following:

THEOREM 5.1. *Let $f(z)$ be analytic on a closed, limited point set C, whose complement is simply connected. If for every n there exists a polynomial $P_n(z)$ of degree n such that*

$$|f(z) - P_n(z)| < \frac{M}{n^{\alpha+1}\rho^n}, \quad \alpha > 0, \rho > 1,$$

on C, then $f(z)$ has derivatives of all orders $\alpha' < \alpha$ on C_ρ.

An approximate converse is:

THEOREM 5.2. *Let $f(z)$ be analytic interior to C_ρ, continuous in the closed region, and let $f(z)$ have a bounded generalized derivative of order $\alpha > 0$ on C_ρ. Then there exist polynomials of respective degrees n such that*

$$|\bar{f}(z) - P_n(z)| < \frac{M \log n}{n^\alpha \rho^n}, \text{ on } C.$$

[1] J. L. Walsh, *Math. Ann.*, **96**, 430–436 and 437–450 (1926).
[2] P. Montel, *Bull. Soc. Math. France*, **46**, 151ff. (1919).
[3] J. Liouville, *Jour. l'Ecole Polytechnique*, **13**, 1–69 (1832); B. Riemann, *Werke*, 331–344.
[4] S. Bernstein, *Leçons Propriétés Extrémales, Paris*, 44–46 (1926).
[5] O. D. Kellogg, *Trans. Amer. Math. Soc.*, **13**, 109–132 (1912).
[6] W. Seidel, *Math. Ann.*, **104**, 182–243 (1931).
[7] W. E. Sewell, *Bull. Amer. Math. Soc.*, **41**, 111–118 (1935).
[8] Hardy and Littlewood, *Math. Zeit.*, **27**, 565–606 (1927–28).
[9] E. Lindelöf, *Acta Societatis Scientiarum Fennicae*, **46**, No. 4, 1–35 (1915).

ABSTRACT CRITICAL SETS

BY MARSTON MORSE AND GEORGE B. VAN SCHAACK

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

Communicated April 12, 1935

This paper is a summary of various results to be published in full at a later time.

PART I. THE GENERAL THEORY. We are concerned with an arbitrary metric space R and a single-valued function $f(P)$ of the point P on R. We suppose that f is continuous and admits a finite absolute minimum. If c is a value assumed by f the domain $f \leqq c$ shall be compact.

Critical points of f are defined as follows: Let P be a point at which $f(P) = c$ and let H be the set of points Q on R at which $f(Q) = c$ and which

are at a distance at most $2e$ from P. Let τ be a variable which ranges on the interval

$$c - \eta \leqq \tau \leqq c + \eta, \tag{1}$$

where η is a positive constant. Let $H \times \tau$ represent the product of H and the interval (1). The point P is termed *ordinary* if for any sufficiently small e and η there exists a homeomorphism T between a neighborhood N of P and $H \times \tau$ with the following property. If the point X on N corresponds under T to the pair (Q, τ) on $H \times \tau$, $f(X) = \tau$. Any point of R which is not ordinary is termed *critical*.

The set of all critical points at which $f = c$, c a constant, is shown to be closed. A *critical set* is defined as a closed set of critical points on which f equals a constant c and which is at a positive distance from other critical points at which $f = c$. Let σ be a critical set and VW an ordered pair of neighborhoods of σ. Indicating closures by adding bars we suppose that $\overline{W} < V$ and that $\overline{V} - \sigma$ contains no critical points at which $f = c$. We shall have frequent occasion to use the phrase "corresponding to an ordered pair of neighborhoods VW" of the above sort, and shall use the expression corr VW as an abbreviation for that phrase.

The definitions of the type number of σ require the introduction of a number of preliminary terms. Let V_c be the set of points on V at which $f < c$.

By a *spannable k-cycle corr VW* we shall mean a k-cycle on W, below c, ~ 0 on W, but not ~ 0 on V_c.

By a *critical k-cycle corr VW* we shall mean a k-cycle on $W + V_c$, not $\sim$ on V to a k-cycle on V_c.

We shall deal with sums of cycles mod 2. Let A, be a class of k-cycles on R which does not include the null cycle. By a *submaximal* set M of cycles of A is meant a subset of elements of A, every proper sum of which belongs to A. The cardinal number r of the cycles in M will be called the *count* of M. The set M will be termed a *maximal* set of cycles of A if it is a proper subset of no submaximal set M' of cycles of A. If M is maximal its count r will be termed the count of cycles of A.

By a *family* F of cycles of A will be meant all proper sums of the cycles of a submaximal set M of cycles of A. We say that M defines F. We show that the count of F is the count of any submaximal set of cycles of A defining F.

We shall say that *a property B holds for any sufficiently limited ordered pair of neighborhoods VW* of σ if there exists a fixed neighborhood N of σ and corresponding to any $V < N$ a neighborhood $M(V)$ such that B holds when $V < N$, $W < M(V)$.

Let t_k be a cardinal number. The critical set σ will be said to have a k-th *subtype number* t_k if for any sufficiently limited VW there exists a family F' of spannable $(k\text{-}1)$-cycles corr VW and a family F'' of critical

k-cycles corr VW such that the count of F' plus the count of F'' equals t_k. The cardinal number of the class of all k-th subtype numbers of σ will be called the k-th *type number* of σ.

Excess and Absolute Type Numbers.—A constant b will be termed a *homology limit* of a cycle z if z lies on $f < b$ and is non-bounding on $f < b$. A cycle which is bounding on R but possesses a homology limit b will be termed an *excess* cycle. Let f_c denote the domain $f < c$. By a *new k-cycle corr VW* is meant a k-cycle on $W + f_c$, not homologous on $V + f_c$ to a cycle on f_c. By a *newly-bounding* (k-1)*-cycle corr VW* is meant a spannable (k-1)-cycle corr VW, not ~ 0 on f_c. If corresponding to sufficiently limited neighborhoods VW there exist families of excess new k-cycles and newly-bounding (k-1)-cycles corr VW of counts α and β, respectively, $\alpha + \beta$ will be called a k-th *excess subtype number* of σ.

Let z be a k-cycle which is non-bounding on R. The greatest lower bound of constants a such that z is homologous on R to a cycle on $f < a$ will be called the *inferior level* of z. By an *absolute* new k-cycle corr VW is meant a new k-cycle which is non-bounding on R and whose inferior level is greater than or equal to the given critical value c. The count γ of a family of absolute new k-cycles corr VW will be termed a k-th *absolute subtype number* of σ.

We show that $\alpha + \beta + \gamma$ is a k-th subtype number of σ.

The cardinal number of the class of all k-th excess (absolute) subtype numbers of σ will be called the k-th *excess* (*absolute*) *type number* of σ.

Let t_k, r_k, e_k be respectively the k-th type number, absolute type number and excess type number of σ. If two of these numbers are finite the third is finite and $e_k + r_k = t_k$. Moreover, t_k is infinite if either e_k or r_k is infinite. In this connection we prove the following theorem.

THEOREM 1. *Let σ be a critical set which is at a positive distance from critical points not on σ. If σ possesses a neighborhood which is coverable by a complex, all but a finite number of its type numbers are null.*

This theorem applies at once to functions which are defined in euclidean space, which are differentiable or merely continuous, and which possess an isolated critical set.

By a *complete* critical set is meant the set of all critical points at which f assumes a critical value c.

The following theorem is fundamental. It holds regardless of whether the critical values are isolated or not, or whether the respective critical sets have infinite type numbers or not.

THEOREM 2. *Let (σ) be an ensemble of complete critical sets with distinct critical values, and with finite excess subtype number sums m_0, m_1, ... There exists a set $(\sigma') > (\sigma)$ of complete critical sets with distinct critical values and with finite k-th excess subtype number sums $M_k \geqq m_k$ such that the following infinite set of (excess) relations holds:*

$$\begin{aligned} M_0 &\geqq 0 \\ M_0 - M_1 &\leqq 0 \\ M_0 - M_1 + M_2 &\geqq 0 \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{2}$$

The following theorem is valid even if the connectivities of R are infinite.

THEOREM 3. *Let F be a finite family of k-cycles which are non-bounding on R. The complete critical sets at the respective inferior levels of the cycles of F have absolute k-th type numbers which sum to at least the count of F. If the k-th connectivity R_k of R is finite the k-th absolute type number sum of f is R_k. If R_k is infinite this sum is infinite.*

The preceding theorem is concerned with the topologically necessary critical sets. The following theorem is concerned with the critical sets in excess of those topologically necessary. The numbers E_k appearing therein may be finite even when the connectivities R_k are infinite.

THEOREM 4. *Let E_k be the k-th excess type number sum of f. If the numbers $E_0, E_1, \ldots$ are finite they satisfy the excess relations (2). If the numbers $E_0, \ldots, E_r$ are finite they satisfy the first $r+1$ excess relations.*

THEOREM 5. *Let T_k be the k-th type number sum of f and R_k the k-th connectivity of R ($k = 0, 1, \ldots$). If T_k and R_k are finite the numbers $M_k = T_k - R_k$ satisfy the excess relations. If T_k and R_k are finite for $k = 0, \ldots, r$ the numbers $M_0, \ldots, M_r$ satisfy the first $r+1$ excess relations.*

PART II. THE CASE OF REGIONS ON REGULAR MANIFOLDS. The preceding theory applies in particular to a function defined on a regular manifold M. It will also apply if the metric space R is a closed region on M. As a function on R, f will in general have critical points on the boundary of R which are not critical points of f considered as a function on M. This follows from our definition of critical points and in the general case has the great advantage that it enables us to proceed without a separate analysis of the boundary and the boundary conditions satisfied by f. If, however, the boundary of R is a set of regular manifolds it is possible to give a more intimate analysis of the critical sets of f on the boundary and their relation to the critical sets of f within R.

The theory of functions on regular regions on regular manifolds is not essentially different from the theory of functions on regular regions of a euclidean space. In Part II we shall accordingly deal with a real single-valued function $f(x_1, \ldots, x_n)$ defined on a limited, open n-dimensional region S of euclidean n-space. We shall say that the function f is of class C^1L on S if it is of class C^1 on S and if its first partial derivatives satisfy Lipschitz conditions neighboring each point of S. In Part II we shall assume that f is of class C^1L on S.

For the moment let us term critical points, sets or type numbers as defined in Part I *abstract* critical points, sets or type numbers, and those

defined in the Colloquium Lectures (Morse) *ordinary*. An abstract critical point of $f(x)$ is an ordinary critical point of $f(x)$, but an ordinary critical point of $f(x)$ is not always an abstract critical point of $f(x)$. An abstract critical set σ' of $f(x)$ is a proper or an improper subset of an ordinary critical set σ. If σ is a given ordinary critical set, the set of abstract critical points of σ forms an abstract critical set σ' which can be shown to have the same type numbers as σ. If σ is an ordinary isolated critical set its type numbers in accordance with Theorem 1 are finite. We make use of this fact throughout Part II. For the remainder of Part II we shall refer to critical sets and type numbers in the ordinary sense and shall omit the word ordinary.

Admissible Functions.—Let Σ be an open subregion of S whose closure $\bar{\Sigma}$ lies on S and whose boundary B is a closed point-set consisting of a finite number of connected, regular, non-intersecting $(n-1)$-spreads of class C^3. Let f_ν denote the directional derivative of f on the normal to B in the sense that leads from points on Σ to points not on Σ. Let f° be the function defined by f on B. Let the existence and type numbers of critical sets of f° be determined in terms of f° as a function of the parameters of B.

The function f will be termed *admissible on* Σ if it satisfies the following conditions.

I. *The function f shall be of class C^1L on an open region containing $\bar{\Sigma}$ and shall have only a finite number of critical values on Σ.*

II. *The function f shall be of class C^3 neighboring B and shall have no critical points on B. The boundary function f° shall have a finite number of critical values.*

The theorem here is as follows.

THEOREM 6. *Let R_i be the i-th connectivity of $\bar{\Sigma}$. Let M_i be the sum of the i-th type numbers of the critical sets of f on Σ and of the boundary function f° defined by f at the points on the boundary of Σ at which $f_\nu < 0$. The numbers $M_i - R_i$ satisfy the first $n+1$ excess relations* (2), *the $(n+1)$-st relation becoming an equality.*

PART III. GROUP THEORY ASPECTS. Certain concepts of the theory in Part I may be developed from the point of view of group theory as follows.

Let $G = A/B$ be the quotient of groups A and B of cycles in which the operation is addition mod 2. The elements of G are classes of cycles, two cycles belonging to the same class if their sum mod 2 belongs to the group B, the zero-element of G. Any class which is not the zero-element of G will be termed a *proper* class. A cycle belonging to a class C will be termed a *representative* of C.

Let VW be an arbitrary pair of neighborhoods of the critical set σ of which $\overline{W} < V$. Let A be the group of k-cycles on W below c, ~ 0 on W, and B the subgroup of cycles ~ 0 on V below c. We term A/B the k-th

spannable group corr VW. A representative of one of its proper classes is a spannable k-cycle corr VW.

Let A be the group of k-cycles on $W + V_c$, and B the subgroup of cycles homologous on V to cycles below c. We term A/B the k-th *critical group corr VW.*

Let f_c denote the domain $f < c$. Let A be the group of k-cycles on $W + f_c$ and B the subgroup of cycles homologous on $V + f_c$ to cycles below c. We term A/B the k-th *new group corr VW.*

Let A be the k-th spannable group corr VW. The elements of A are classes of k-cycles. Let B be the subgroup of A whose classes are represented by cycles which bound below c. We term A/B the k-th *newly-bounding group corr VW.*

Let A be the group of all k-cycles bounding on R, and B the subgroup of cycles without homology limits. We term A/B the k-th *excess group.*

Let A be the group of k-cycles on $W + f_c$ which are bounding on R, and B the subgroup of cycles which are homologous on $V + f_c$ to cycles below c. We term A/B the k-th *excess new group corr VW.*

Let A be the homology group of R and B the subgroup of A whose classes are represented by cycles with inferior levels less than c. We term A/B the k-th *absolute new group corr VW.*

The groups introduced in Part III correspond to the families introduced in Part I. The families seem to offer the more concise characterization of the categories of cycles needed, while the group theory representation throws a new light on their structure.

Morse, "The Calculus of Variations in the Large," *Amer. Math. Soc. Colloquium Publications*, **18,** New York (1934).

Birkhoff, G. D., and Hestenes, M. R., "Generalized Minimax Principle in the Calculus of Variations," these PROCEEDINGS, **21,** 96–99 (1935).

ON CONVERGENCE FACTORS FOR SERIES SUMMABLE BY NÖRLUND MEANS

BY CHARLES N. MOORE

INSTITUTE FOR ADVANCED STUDY

Communicated April 15, 1935

All methods for summing divergent series that have been widely used in practice may be classified as methods of means or methods of convergence factors. In methods of means the value of the series is defined as the limit of weighted means of the partial sums of the series. In methods of convergence factors the series is rendered convergent over a certain range of values of a parameter by the introduction of a set of functions

of the parameter as factors of its terms. Thus the series with convergence factors defines a function over this range of values of the parameter. When this function approaches a limit as the parameter approaches a certain value, while at the same time all the convergence factors approach unity, this limit is defined as the value of the series.

The first use of the term convergence factor in connection with a method for summing certain classes of divergent series dates back to 1907.[1] Subsequently the term was found convenient by other writers.[2] It was used systematically in Chapter VI of the second edition of Borel's, *Leçons sur les séries divergentes*, but without any clear indication of its origin.

The beginning of the modern theory of summability of series may be found in a paper by Frobenius,[3] which is essentially the proof of a convergence factor theorem. That is to say, it is shown that a certain set of convergence factors will sum a series to the same value as that obtained by a certain mean value process. Subsequently many other convergence factor theorems have been obtained by various writers. They have dealt in the main with series summable by Cesàro means or the equivalent Hölder means.

A natural generalization of Cesàro means is found in the so-called Nörlund means.[4] Given an infinite set of complex constants, $c_0 \neq 0$, c_1, c_2, ..., c_n, ..., we set

$$C_n = c_0 + c_1 + \ldots + c_n. \tag{1}$$

For any series Σu_n, we define

$$\sigma_n = S_n/C_n = (c_n s_0 + c_{n-1} s_1 + \ldots + c_0 s_n)/C_n, \tag{2}$$

where $s_n = u_0 + u_1 + \ldots + u_n$. If σ_n approaches a limit, σ, as n becomes infinite, we say that the series Σu_n is summable $(N; c)$ to σ. To be widely useful in the summation of series it is essential that the Nörlund means considered evaluate all convergent series to the proper sum. For this purpose the c's should satisfy the so-called regularity conditions

$$\sum_{i=1}^{i=n} |c_i| < A\,|C_n|, \qquad [c_n/C_n] \longrightarrow 0 \text{ as } n \longrightarrow \infty, \tag{3}$$

where A is a positive constant.

The purpose of the present note is to supply necessary and sufficient conditions that a set of convergence factors will sum to the same value any series summable by Nörlund means of a certain type. The result obtained includes as special cases a variety of previous results on sufficient, and necessary and sufficient conditions for convergence factors in series summable by Cesàro means.[5]

We consider a set of complex constants, c_n, for which the series

$$C_0 + C_1 z + C_2 z^2 + \ldots + C_n z^n + \ldots, \tag{4}$$

where the C's are defined as in (1), converges within the unit circle, has no vanishing points there and diverges for $z = 1$, as does also $\sum c_n z^n$. We set

$$1/(C_0 + C_1 z + C_2 z^2 + \ldots) = a_0 + a_1 z + a_2 z^2 + \ldots, \qquad (5)$$

and we further assume that

$$\Sigma n\,|a_n| < \infty, \qquad (6)$$

We now introduce a set of functions, $f_n(\alpha)$ $[n = 0, 1, 2, \ldots]$, defined over a set of values $E(\alpha)$, having a limit point α_0, not of the set. We require first that

$$|C_n f_n(\alpha)| < M(\alpha) \qquad [E(\alpha);\ n = 0, 1, 2, \ldots], \qquad (A)$$

$M(\alpha)$ being a positive function of α defined over $E(\alpha)$. Then we define

$$Lf_n(\alpha) = a_0 f_n(\alpha) + a_1 f_{n+1}(\alpha) + a_2 f_{n+2}(\alpha) + \ldots, \qquad (7)$$

the convergence of the series on the right being assured by (6), (3) and (A). We require further that

$$\sum_{n=0}^{\infty} |C_n\, Lf_n(\alpha)| < K(\alpha) \qquad [E(\alpha)], \qquad (B)$$

$K(\alpha)$ being a positive function of α defined over $E(\alpha)$.

We can now establish the following theorem:

THEOREM I. *If the series Σu_n is summable $(N; c)$, where the c's satisfy the regularity conditions* (3) *and are such that* (6) *holds, then the necessary and sufficient conditions that the series*

$$\Sigma u_n f_n(\alpha) \qquad (8)$$

converge over $E(\alpha)$ are (A) and (B).

In order that the function $F(\alpha)$, defined by the series (8), may approach the value to which the series Σu_n is summable $(N; c)$ as $\alpha \longrightarrow \alpha_0$, we must make further restrictions on the convergence factors $f_n(\alpha)$. These supplementary conditions are as follows:

$$\lim_{\alpha \to \alpha_0} Lf_n(\alpha) = 0 \qquad (n = 0, 1, 2, \ldots), \qquad (A_1)$$

$$\sum_{n=0}^{\infty} |C_n\, Lf_n(\alpha)| < K \qquad [E'(\alpha)], \qquad (B_1)$$

$$\lim_{\alpha \to \alpha_0} f_n(\alpha) = 1 \qquad (n = 0, 1, 2, \ldots), \qquad (C)$$

where $E'(\alpha)$ is a certain sub-set of $E(\alpha)$, having also α_0 as a limit point, and K is a positive constant.

The main convergence factor theorem may then be stated as follows:

THEOREM II. *If the series Σu_n is summable $(N; c)$ to σ, the c's satisfying the conditions of Theorem I, the necessary and sufficient conditions that the series (8) should converge in $E(\alpha)$ and approach σ as a limit as α approaches α_0, are (A), (A_1), (B_1), and (C).*

[1] C. N. Moore, "On the Introduction of Convergence Factors into Summable Series and Summable Integrals," *Trans. Amer. Math. Soc.*, **8**, 299–330 (1907).

[2] G. H. Hardy, *Proc. London Math. Soc.* (2), **6**, 264 (1908); S. Chapman, *Proc. London Math. Soc.* (2), **9**, 369–409 (1911). Cf. sections 13–17.

[3] *Jour. Math.*, **89**, 262–264 (1880).

[4] Nörlund, "Sur une application des fonctions permutables," *Lunds Universitets Års-skrift*, N. F., **16**, No. 3 (1920). Cf. also Woronoi-Tamarkin, *Ann. Math.* (2), **33**, 422–428 (1932).

[5] Bromwich, *Math. Ann.*, **65**, 350–369 (1908); Chapman, loc. cit., footnote 2; W. A. Hurwitz, *Bull. Amer. Math. Soc.* (2), **28**, 156 (1922).

EFFECTIVE EQUALITY AND EFFECTIVE IMPLICATION IN FORMAL LOGIC

BY EDWARD V. HUNTINGTON

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

Communicated March 26, 1935

Introduction.—The development of an abstract mathematical theory may be described, in the last analysis, as a process of writing down, one after another, a series of expressions—this process involving the motor activity of some human agent.

Every such theory deals with a specified "universe of discourse," usually comprising a set of objects, which may be called elements of a class K, and certain specified operations and relations among these elements. The development of the theory starts with a set of "postulates" agreed upon in advance and then proceeds to deduce "theorems" as logical consequences of these postulates.

The postulates are usually of two kinds: (1) specific rules of procedure which authorize one to proceed from expressions already established to new expressions; and (2) a list of initially accepted expressions, agreed upon in advance. In applying the specific rules of procedure, use is also made of certain general rules of procedure which are often included under the head of intuitive logic.

Such a theory is "abstract" because the symbols with which it deals are so-called "variables," capable of taking on any interpretations which are consistent with the postulates. The interest of an abstract mathematical

theory usually depends upon the interest attaching to some one of the concrete interpretations of that theory.

The purpose of the present paper is to present an abstract mathematical theory which is capable of being interpreted as a part of the calculus of propositions—the part, namely, which is expressible in terms of the operation of conjunction and the relation of equality, without the use of negation or "truth values."

It will be found that an abstract relation $a < b$ (read: a less than b), can be defined in terms of an abstract operation ab (read: a times b), and an abstract relation $a \cong b$ (read: a approaches b), in such a way that when ab is interpreted as the operation of conjunction ("a and b"), and $a \cong b$ is interpreted as the relation of equality ("a equals b") then $a < b$ becomes the relation of implication (a implies b). [In order to distinguish this relation $a < b$ from "material implication" and "strict implication," it may be called "effective implication."]

A set of independent postulates for ab and $a \cong b$ is given, and all the "general rules of procedure" employed in the proofs of theorems are listed explicitly.

General Rules of Procedure.—In the following general rules of procedure, the letters X, Y, Z denote any expressions. The arrow notation ($\rightarrow$) means that whenever we find that the expression preceding the main arrow has been already established, we are thereupon authorized to write down the expression following the arrow, as a newly accepted expression.

In particular, Rule I means that whenever we find an expression X established on *any line* of our work, we may copy down this same expression X on a new line. Rule II means that if we find X established on *any line*, and Y on *any other line*, we may write down the expression X & Y on a new line. In each of the remaining rules, III–VIII, the meaning is that whenever we find that the expression preceding the main arrow has been established on what is at the moment, the *last line* of our work, then the expression following the arrow may be written down on a new line.

General rules of procedure.

I. $X \rightarrow X$.
II. $(X \text{ and } Y) \rightarrow (X \,\&\, Y)$.
III. $(X \,\&\, Y) \rightarrow X$.
IV. $(X \,\&\, Y) \rightarrow Y$.
V. $[X \,\&\, (X \rightarrow Y)] \rightarrow Y$.
VI. $[(X \rightarrow Y) \,\&\, (Y \rightarrow Z)] \rightarrow (X \rightarrow Z)$.
VII. $[(X \rightarrow Y) \,\&\, (X \rightarrow Z)] \rightarrow [X \rightarrow (Y \,\&\, Z)]$.
VIII. $[(X \,\&\, Y) \rightarrow Z] \rightarrow [X \rightarrow (Y \rightarrow Z)]$.

The following supplementary rules IX–XI are deducible as consequences of Rules I–VIII, and are added at this point for convenience of reference.

IX. $\{[(X \& Z) \rightarrow Y] \& Z\} \rightarrow (X \rightarrow Y)$.
X. $[(X \rightarrow Y) \& Z] \rightarrow [X \rightarrow (Y \& Z)]$.
XI. $X \rightarrow (Y \rightarrow X)$.

Postulates for the System $(K, T, \times, \cong)$.—The following postulates 10–23 serve to define the system $(K, T, \times, \cong)$ which is considered in the present paper. Postulates 10–13 are *descriptive postulates*, which describe the "universe of discourse" indicated by the notation $(K, T, \times, \cong)$; Postulates 14–19 are *rules of procedure*, by which we are authorized to proceed from expressions already established to new expressions; Postulate 20 is the *rule of substitution;* and Postulates 21–23 give the *initially accepted expressions* of the system. (Examples of specific systems $(K, T, \times, \cong)$ will be given below.)

Descriptive postulates.

10. K denotes some specified (non-empty) class of elements, $a, b, c, \ldots$.
11. $a \times b$, or simply ab, denotes an element of K uniquely determined by the elements a and b according to some specified binary rule of operation.
12. $a \cong b$ denotes a relation between the elements a and b, which is or is not valid according to some specified rule of relationship.
13. a in T denotes that the element a belongs to a subclass T characterized by some specified property.

Rules of procedure.

14. $(a \cong b) \rightarrow (b \cong a)$.
15. $[(a \cong b) \& (b \cong c)] \rightarrow (a \cong c)$.
16. $(a \cong b) \rightarrow [ac \cong bc]$.
17. $[(a \text{ in } T) \& (a \cong b)] \rightarrow (b \text{ in } T)$.
18. $[(a \text{ in } T) \& (b \text{ in } T)] \rightarrow (ab \text{ in } T)$.
19. $(ab \text{ in } T) \rightarrow (b \text{ in } T)$.

Rule of substitution.

20. In these postulates 14–19 and also in the following postulates 21–23, the letters a, b, c are *variables* standing for any elements of K; that is, the letter a may be "replaced throughout" by any element of K; the letter b may be "replaced throughout" by any element of K; and the letter c may be "replaced throughout" by any element of K (that is, by any letter representing such an element).

Initially accepted expressions.

21. $ab \cong ba$.
22. $(ab)c \cong a(bc)$.
23. $a \cong aa$.

Definition of $a < b$.—In order to simplify the appearance of the theorems

which follow from these postulates, we introduce the following abbreviation:

24. Definition. $(a < b) = (a \cong ab)$. (Read: a less than b.)

Here the symbol "$=$" indicates that the two expressions between which it stands are "equal by definition"—that is, either of them may be replaced by the other *at any point* in any expression of the system.

Theorems.—The following theorems are deducible from Postulates 10–23 (and Definition 24) by the aid of the general rules of procedure I–VIII.

Theorems containing the arrow ($\rightarrow$) are additions to the list of "rules of procedure"; theorems not containing the arrow are additions to the list of "accepted expressions."

30. $a \cong a$. ("Self-identity.")
31. $(a \cong b) \rightarrow [(a < b) \,\&\, (b < a)]$. ("Mutual implication.")
32. $[(a < b) \,\&\, (b < a)] \rightarrow (a \cong b)$.
33. $ab < a$. ("Simplification.")
34. $ab < b$.
35. $[(a \text{ in } T) \,\&\, (a < b)] \rightarrow (b \text{ in } T)$. ("Inference.")
36. $[(a < b) \,\&\, (b < c)] \rightarrow (a < c)$. ("Syllogism.")
37. $[(a < b) \,\&\, (a < c)] \rightarrow [a < bc]$. ("Composition.")
38. $(a < b) \rightarrow (ac < b)$.

Propositions Not Belonging to the Present Theory.—It will be observed that Theorems 33–37 are roughly analogous to the general rules of procedure III–VII; but the following propositions, roughly analogous to IX, X, XI, cannot be deduced from Postulates 10–23;

[A.] $[(ac < b) \,\&\, (c \text{ in } T)] \rightarrow (a < b)$.
[B.] $[(a < b) \,\&\, (c \text{ in } T)] \rightarrow (a < bc)$.
[C.] $(c \text{ in } T) \rightarrow (a < c)$.

These theorems and propositions illustrate the similarities and the differences between the symbol "$<$" and the symbol "$\rightarrow$." The most striking difference lies in the fact that the "$\rightarrow$" can appear on different "levels" within the same formula, while the "$<$" cannot.

The properties of the symbol "$<$" as here defined differ from the properties of "material implication" ($\supset$) as developed by Whitehead and Russell, but agree with the properties of "strict implication" ($-\!\!<$) as developed by C. I. Lewis (at least in so far as Lewis' properties are expressed in terms of "conjunction" and "equivalence" alone, without the use of "negation").

Examples of Systems $(K, T, \times, \cong)$ *Which Satisfy All the Postulates.*—The following examples of systems $(K, T, \times, \cong)$ satisfy all the postulates 10–23.

EXAMPLE A.

K means the class of business corporations, $a, b, c, \ldots$, each corporation being subject to taxation.

ab means the corporation formed by a merger of the corporations a and b.

$a \cong b$ means that the tax paid by a is the same as the tax paid by b.

a in T means that a is a politically favored corporation which has to pay only a nominal tax of one dollar.

Here all the postulates 10–23 are seen to be satisfied, and the relation $a < b$, defined as meaning $a \cong ab$, may be interpreted as "a controls b."

EXAMPLE B.

K means the class of "propositions."

ab means the proposition "a and b."

$a \cong b$ means that the propositions a and b are "equal in effect."

a in T means that the proposition a is "true."

Here all the postulates 10–23 may be said to be satisfied, and the relation $a < b$, defined as meaning $a \cong ab$, may be interpreted as "a implies b"—that is, proposition a carries proposition b with it in the sense that adjoining b to a does not add anything to the effect that would be produced by a alone.

In view of the importance of Example B, any system $(K, T, \times, \cong)$ which satisfies the postulates 10–23 may be called a system of *effective implication*. It must not be overlooked, however, that the following "dual" system satisfies the same postulates:

EXAMPLE C.

K means the class of "propositions."

ab means the proposition "a or b."

$a \cong b$ means that the propositions a and b are "equal in effect."

a in T means that the proposition a is "false.'

Here again all the postulates 10–23 may be said to be satisfied, and the relation $a < b$, defined as meaning $a \cong ab$, may be interpreted as "b implies a."

There is nothing in the abstract postulates themselves by which the "and-true" interpretation can be distinguished from the "or-false" interpretation.

EXAMPLE D.

K means the class of eight numbers, 1, 2, 3, 4, 5, 6, 7, 8.

ab means the number given by the adjoining multiplication table.

$a \cong b$ means that a dot is found opposite a and underneath b in the adjoining relation table.

a in T means that a belongs to the subclass $T = 1, 2$.

(The relation table for $a < b$ is added for reference.) $(a < b) = (a \cong ab.)$

ab	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	2	4	4	6	6	8	8
3	3	4	3	4	7	8	7	8
4	4	4	4	4	8	8	8	8
5	5	6	7	8	5	6	7	8
6	6	6	8	8	6	6	8	8
7	7	8	7	8	7	8	7	8
8	8	8	8	8	8	8	8	8

$a \cong b$	1	2	3	4	5	6	7	8
1	.							
2		.						
3			.					
4				.				
5					.			
6						.		
7							.	
8								.

$a < b$	1	2	3	4	5	6	7	8
1	.							
2	.	.						
3	.		.					
4	.	.	.	.				
5	.				.			
6	.	.				.		
7	.		.				.	
8	.	.	.	.	.	.	.	.

This finite system satisfies all the postulates 10–23, and thus proves their consistency in the usual sense.

PROOFS OF INDEPENDENCE.

Each of the following systems $(K, T, \times, \cong)$ violates the like-numbered postulate, but satisfies all the other postulates of the list 10–23.

EXAMPLE 14.

K means 1, 2, 3, 4.
ab means the number given by the table.
$a \cong b$ means the relation given by the table.
a in T means $a = 1$ or 2.

Here 14 fails when $a = 1$, $b = 2$.

ab	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

$a \cong b$	1	2	3	4
1	.	.		
2		.		
3			.	.
4				.

EXAMPLE 15.

Same as Example D except that $a \cong b$ is given by a new table.

EXAMPLE 16.

Same as Example D except that $a \cong b$ is given by a new table.

For Ex. 15

$a \cong b$	1	2	3	4	5	6	7	8
1	.							
2		.						
3			.	.				
4			.	.	.	.	.	.
5					.			.
6						.		.
7							.	.
8					.	.	.	.

Here 15 fails
$a = 3$, $b = 4$, $c = 5$.

For Ex. 16

$a \cong b$	1	2	3	4	5	6	7	8
1	.							
2		.						
3			.	.	.	.	.	.
4			.	.				
5			.		.			
6			.			.		
7			.				.	
8			.					.

Here 16 fails when
$a = 6$, $b = 3$, $c = 6$.

EXAMPLE 17.

Same as Example A, except that "a in T" means "a is honest."

EXAMPLE 18.

Same as Example A, except that "a in T" means "a is poor."

EXAMPLE 19.

Same as Example A, except that "a in T" means "a is rich."

EXAMPLE 21.

K means the class of four numbers, 1, 2, 3, 4.
ab means the number b.
$a \cong b$ means that a and b are equal.
a in T means that a is 1 or 2.

Here 21 obviously fails.

EXAMPLE 22.

K means the class of four numbers, 1, 2, 3, 4.
ab means the number given by the table.
$a \cong b$ means that a and b are equal.
a in T means that a is 1 or 2.

Here 22 fails, since $(23)4 = 44 = 4$ while $2(34) = 24 = 3$.

ab	1	2	3	4
1	1	2	4	3
2	2	2	4	3
3	4	4	3	4
4	3	3	4	4

EXAMPLE 23.

K means the class of four numbers, 1, 2, 3, 4.
ab means the number given by the table.
$a \cong b$ means that a and b are equal.
a in T means that a is 1 or 2.

Here 23 fails when $a = 2$.

ab	1	2	3	4
1	1	2	3	3
2	2	1	3	3
3	3	3	3	3
4	3	3	3	4

These examples establish the independence of Postulates 14–19 and 21–23 in the list of Postulates 10–23. Independence proofs for Postulates 10–13 and 20 are readily constructed. Hence no one of the Postulates 10–23 can be omitted without changing the character of the system.

A THEOREM ON EXTREMALS. I

BY S. LEFSCHETZ

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

Communicated April 15, 1935

In the present paper we state and outline a proof of a theorem on conjugate points and foci of extremals which leads in the most direct way possible to Morse's fundamental theorem on their type numbers.[1] In the simplest case our proposition reduces to the criterion of Bliss,[2] and as an important special case we have a result of Hahn,[3] while the general theorem is implicit in Morse's work. As a noteworthy by-product we have completed Kneser's partial results linking envelopes of extremal pencils with their conjugate points or foci.[4]

For the sake of convenience we assume analyticity throughout, but well standardized methods lead from there to say class C'''.

1. Consider a positive regular problem of the Calculus of Variations with analytical integrand attached to an absolute topological analytical M_{n+1}, and let g be an indefinitely extended extremal. We suppose the situation so chosen that g is the x-axis, both being oriented alike. The integral is

$$J(A, B) = \int_A^B f(x, y, y')dx,$$

where y, y' stand for y_i, y_i', $i = 1, 2, \ldots, n$.

Let A, B be two points on g (A before B) and M_n, M_n^* two analytical manifolds (terminal manifolds) regular at A, B and transversal to g without contact in these points. Their successive foci will be described respectively by A^{*i}, B^{*i}, where say A^i follows A^{i-1}, and $A^\circ = A$. The pencils of extremals transversal to M_n and M_n^* will be denoted by $\{\gamma\}$ and $\{\gamma^*\}$, and g is a member of both.

Consider now the arc $A'B'$ intercepted on a γ^* very near g. If z_i are coördinates on M_{n-1} about A we have

$$J(A'B') = J(AB) + \phi_2(z) + \phi_3(z) + \ldots, \qquad (1)$$

(ϕ_i a form of degree i, ϕ_2 non-degenerate).

We shall denote by h the *index* of ϕ_2, or number of negative squares in its reduced form. On the other hand the envelope E of $\{\gamma\}$ has with g in A^i, $i > 0$, a contact whose order is the *index* of this focal point. In particular we denote by k the index of A^1.

We may now state our

THEOREM. *If $B^{-1}AA^1BB^1A^2$ follow in the order named then index ϕ_2 = index A^1 ($h = k$).*

2. *Outline of Proof.*—We show that: (*a*) There is an analytical l-cell, $l = n - k$, of elements $A'B'$ for which $J(A'B') > J(AB)$ (except when $A'B' = AB$); (*b*) There is a similar k-cell for which the sign is reversed and which is not tangent to the first. An important feature of the proof is that we give a precise description of the two different cells.

Let a_{ii} be parameters in $\{\gamma\}$, so chosen that g is represented by (0), and let $x = 0$ at A^1. About g we have for γ equations analytical in all variables

$$y_i = y_i(x, a) \tag{2}$$

$$rank \left\| \left(\frac{\partial y_i}{\partial a_j} \right)_{x=0} \right\| = n - k = l. \tag{3}$$

We now assume that $k < n$, hence $l > 0$. Then

$$y_i = \lambda_{ij}(x)a_j + \ldots, \tag{4}$$

where we may furthermore suppose the a's so chosen that $\lambda_{ij}(0) = 0$ for $j > l$, and that at A^1, and hence on AB^1, closed,

$$l = \text{rank} \parallel \lambda_{ij}(x) \parallel, \quad j = 1, \ldots, l. \tag{5}$$

As a consequence, on setting every $a_{l+j} = 0$ in (4) there is determined in M_n a certain M_l. The set of all γ^*'s meeting M_l determines then an $M_{l+1} \supset g$. We have then a certain point-function $F(P)$ constant on the extremal-parallels to M_n and in particular on M_l of M_{l+1} and it is analytical on M_{l+1}. Owing to (5) and the fact that $\{\gamma^*\}$ is a field up to B^1, $F(P)$ can be "propagated" along g clear up to B^1, and yields thereon extremal-parallels (intersections with those on M_{n+1}) which reach B^1, and are regular along g on AB closed. The transversal extremals on M_{l+1} determine a (relative) field on M_{l+1} which $\supset g$ and reaches almost B^1, and for which A^1 ceases to be a focus. Since AB is in the field, we have at once from the criterion of Weierstrass that for $A' \subset M_l$, or $A'B' \subset M_{l+1}$, $J(A'B') > J(AB)$, which proves (*a*).

3. Reversing the situation we now choose $a_i = 0$ for $i \leqq l$ so that we have an analytical extremal γ represented by equations

$$y_i = \lambda_{ij}(x)a_j + \ldots \qquad j > l, \ \lambda_{ij}(0) = 0. \tag{6}$$

By a familiar argument $\dfrac{\partial^2 y_i(o, a)}{\partial x \, \partial a_j} \neq 0$, and hence $\lambda_{ij}(x) = \mu_{ij} \cdot x$, $\mu_{ij} \neq 0$. Since the a's are suitable coördinates on M_n, to prove (b) we merely need to show that if the point A' is on the trace M_k of the preceding extremals, and $A' \neq A$, then $J(A'B') < J(AB)$. Since $\{\gamma^*\}$ has no focus on AB it will be sufficient to prove that this inequality holds for some arc $A'B''$

issued from A' and very near AB. If t is any linear combination of the a's in (6)

$$y_i = t(\mu_i \cdot x + \mu_i^1(x)t + \ldots), \mu_i \neq 0 \qquad (7)$$

represents an arbitrary γ of (6) which has an envelope-curve Γ. If Γ is a point (i.e., $k = n$) our result is immediate. If not, it is tangent in A^1 to g. When Γ has arcs on both sides, it is shown that when γ touches Γ to the left of A^1, i.e., when t is either > 0 or < 0, the requirement holds. By an obvious property of quadratic forms this will still be true when t has the opposite signs. Thus the only case to be disposed of is the one left open by Kneser, namely, *when* Γ *has in* A *a cusp pointed to the left.* This is the "foyer en talon" of French authors and it is treated as follows. Let Δ be one of the arcs issued from A^1 and let $\bar{g}$ be a γ very near g which touches Δ in A^1. As transversality in B to M_n^* does not matter here, we may find a $\bar{\gamma}$ touching Δ slightly beyond A^1 and behaving as required relative to $\bar{g}$ and to a slight deform $\overline{M}_n^*$ of M_n^*. We may then replace our steps without weakening the basic inequality, from which the ultimate theorem follows.

4. The second part may also be derived analytically as follows: We may here replace M_n and M_n^* by extremal-parallel manifolds as near A^1 as we please. If we set $f_{y_i' y_j'}(x_0, 0, 0) = \phi_{ij}$, and if $x - x_0 = \xi t + \ldots$ is the equation of the arc of M_n described by A' as t varies, we find by direct integration

$$J(A'B') - J(AB) = \tfrac{1}{2} x_0 \phi_{ij} \mu_i \mu_j t^2 + \ldots < 0$$

for t sufficiently small and $x_0 < 0$, as was to be proved.

[1] See M. Morse, "Calculus of Variations in the Large," *Am. Math. Soc. Coll. Publ.*, **18,** Chs. 3, 7, 8 (1934).

[2] G. A. Bliss, *Math. Ann.*, **58,** 70 (1904).

[3] H. Hahn, *Math. Ann.*, **70,** 110–142 (1911); also J. Rosenberg, *Monatsh. Math. Physic,* **24,** 65–86 (1913). Hahn was the first to consider the effect of the index of a conjugate point on the integral differences and obtained a final result for extremal triangles in an M.

[4] A. Kneser, *Lehrbuch der Variationsrechnung.*

A PROOF OF FREEDOM FROM CONTRADICTION

BY ALONZO CHURCH

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

Communicated March 26, 1935

1. *A System of Logic.*—We take as undefined the symbols, {, }, (,), λ, [,], δ and an infinite list of variables, a, b, c, ... Following previous usage,[1] with minor changes, we define a *formula* to be a finite sequence of undefined symbols, and define the terms *well formed*, *free*, *bound*, by induction, as follows. The symbol δ standing alone is a well-formed formula; any variable $\mathbf{x}$ standing alone is a well-formed formula and the occurrence of $\mathbf{x}$ in it is an occurrence as a free variable in it; if the formulas $\mathbf{M}$ and $\mathbf{N}$ are well formed, $\{\mathbf{M}\}(\mathbf{N})$ is well formed, and an occurrence of a variable $\mathbf{x}$ as a free (bound) variable in $\mathbf{M}$ or in $\mathbf{N}$ is an occurrence of $\mathbf{x}$ as a free (bound) variable in $\{\mathbf{M}\}(\mathbf{N})$; if the formula $\mathbf{R}$ is well formed and contains an occurrence of $\mathbf{x}$ as a free variable in $\mathbf{R}$, then $\lambda\mathbf{x}[\mathbf{R}]$ is well formed, any occurrence of $\mathbf{x}$ in $\lambda\mathbf{x}[\mathbf{R}]$ is an occurrence of $\mathbf{x}$ as a bound variable in $\lambda\mathbf{x}[\mathbf{R}]$, and an occurrence of a variable $\mathbf{y}$, other than $\mathbf{x}$, as a free (bound) variable in $\mathbf{R}$ is an occurrence of $\mathbf{y}$ as a free (bound) variable in $\lambda\mathbf{x}[\mathbf{R}]$.

A formula $\{\mathbf{F}\}(\mathbf{X})$ is abbreviated as $\mathbf{F}(\mathbf{X})$ in all cases where $\mathbf{F}$ is or is represented by a single symbol. A formula $\{\{\mathbf{F}\}(\mathbf{X})\}(\mathbf{Y})$ is abbreviated as $\{\mathbf{F}\}(\mathbf{X}, \mathbf{Y})$, or, if $\mathbf{F}$ is or is represented by a single symbol, as $\mathbf{F}(\mathbf{X}, \mathbf{Y})$. And $\{\{\{\mathbf{F}\}(\mathbf{X})\}(\mathbf{Y})\}(\mathbf{Z})$ is abbreviated as $\{\mathbf{F}\}(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$, or as $\mathbf{F}(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$, and so on. A formula $\lambda\mathbf{x}_1[\lambda\mathbf{x}_2[\ldots\lambda\mathbf{x}_n[\mathbf{R}]\ldots]]$ is as abbreviated as $\lambda\mathbf{x}_1\mathbf{x}_2\ldots\mathbf{x}_n[\mathbf{R}]$. If $\mathbf{F}$ is or is represented by a single symbol, $\{\mathbf{F}\}(\lambda\mathbf{x}[\mathbf{M}])$ is abbreviated as $\mathbf{F}\mathbf{x}[\mathbf{M}]$. Whenever possible without ambiguity, brackets [] are omitted, whether the brackets are the undefined symbols [], or whether they are themselves part of an abbreviation. A dot occurring in a formula stands for omitted bracket extending from the position of the dot forward the maximum distance that is consistent with the formula's being well formed. And when omitted brackets are not replaced by a dot, they are understood to extend forward the minimum distance.

Following Kleene (1934), we reserve heavy type letters to represent undetermined formulas in metamathematical discussions, and adopt the conventions that each heavy type letter shall represent a well-formed formula and each set of symbols standing apart which contains a heavy type letter shall represent a well-formed formula. A formula is said to be in λ-*normal form* if it is well formed and has no part of the form $\{\lambda\mathbf{x}\mathbf{M}\}(\mathbf{N})$. A formula is said to be in *normal form* if it is in λ-normal form and has no part which has the form $\delta(\mathbf{M}, \mathbf{N})$ and which contains no occurrence of any variable as a free variable in $\delta(\mathbf{M}, \mathbf{N})$.

The expression $S^{\mathbf{x}}_{\mathbf{N}}\mathbf{M}|$ is used to stand for the result of substituting **N** for **x** throughout **M**.

We adopt a single formal postulate, $\lambda fx.f(f(x))$, and seven rules of procedure:

I. *To replace any part* $\lambda\mathbf{xR}$ *of a formula by* $\lambda\mathbf{y}S^{\mathbf{x}}_{\mathbf{y}}\mathbf{R}|$, *where* **y** *is any variable which does not occur in* **R**.

II. *To replace any part* $\{\lambda\mathbf{xM}\}(\mathbf{N})$ *of a formula by* $S^{\mathbf{x}}_{\mathbf{N}}\mathbf{M}|$, *provided that the bound variables in* **M** *are distinct both from* **x** *and from the free variables in* **N**.

III. *To replace any part* $S^{\mathbf{x}}_{\mathbf{N}}\mathbf{M}|$ (*not immediately following* λ) *of a formula by* $\{\lambda\mathbf{xM}\}(\mathbf{N})$, *provided that the bound variables in* **M** *are distinct both from* **x** *and from the free variables in* **N**.

IV. *To replace any part* $\delta(\mathbf{M}, \mathbf{N})$ *of a formula by* $\lambda fx.f(f(x))$, *where* **M** and **N** *are in normal form and contain no free variables, and* **M** *conv-I* **N**.

V. *To replace any part* $\delta(\mathbf{M}, \mathbf{N})$ *of a formula by* $\lambda fx.f(x)$, *where* **M** *and* **N** *are in normal form and contain no free variables, and it is not true that* **M** *conv-I* **N**.

VI. *To replace any part* $\lambda fx.f(f(x))$ *of a formula by* $\delta(\mathbf{M}, \mathbf{N})$, *where* **M** *and* **N** *are in normal form and contain no free variables, and* **M** *conv-I* **N**.

VII. *To replace any part* $\lambda fx.f(x)$ *of a formula by* $\delta(\mathbf{M}, \mathbf{N})$ *where* **M** *and* **N** *are in normal form and contain no free variables, and it is not true that* **M** *conv-I* **N**.

We are here using the notation **M** *conv-I* **N** to mean that **N** is obtainable from **M** by a sequence of applications of Rule I. It should be observed that, given any two formulas, **M** and **N**, it is constructively possible to determine whether **M** conv-I **N**.

A sequence of applications of Rules I–VII will be called a *conversion*, a sequence of applications of Rules I–III a λ-*conversion* and a sequence of applications of Rules IV–VII a δ-*conversion*. It will be said that **M** is *convertible* (λ-*convertible*, δ-*convertible*) into **N** if **N** is obtainable from **M** by a conversion (λ-conversion, δ-conversion). And the phrase *is convertible into* will be abbreviated as *conv*. A *reduction* is a conversion, each step of which is an application of one of the rules I, II, IV, V, and in which one and only one step is an application of a rule other than Rule I. The formula **Y** is said to be *a normal form* (*a* λ-*normal form*) *of* the formula **X** if **Y** is in normal form (λ-normal form) and **X** conv **Y** (**X** λ-conv **Y**).

The following theorem may be proved by induction on the length of the formula:

THEOREM I. *If a formula is in normal form no reduction of it is possible.*

The following will be proved in a forthcoming paper:[2]

THEOREM II. *If a formula has a normal form, this normal form is unique to within applications of Rule I, and any sequence of reductions of the formula must* (*if continued*) *terminate in the normal form.*

And from this follows readily:

THEOREM III. *If a well-formed part of a formula has no normal form, the whole formula likewise has no normal form.*

2. *Equality, Negation, Logical Product.*—We introduce the following definitions and abbreviations, where the arrow is to be read, "stands for," or, "is an abbreviation for."

$1 \rightarrow \lambda fx.f(x)$,

$2 \rightarrow \lambda fx.f(f(x))$, and so on, for all the positive integers.

$\sim \rightarrow \lambda x.6 - [\delta(x, 1) + 2\delta(x, 2)]$.

$\sim[\mathbf{P}] \rightarrow \sim(\mathbf{P})$.

$\& \rightarrow \lambda xy.6 - \min(\delta(x, 1) + 2\delta(x, 2), \delta(y, 1) + 2\delta(y, 2))$.

$[\mathbf{P}][\mathbf{Q}] \rightarrow \&(\mathbf{P}, \mathbf{Q})$.

Here the functions denoted by $+$, $-$, min and multiplication, denoted by juxtaposition, as in $2\delta(x, 2)$, are supposed defined as by Kleene. Where necessary to avoid confusion with the abbreviation for $\&(\mathbf{P}, \mathbf{Q})$, multiplication will be denoted by $\times$. If $\mathbf{m}$ and $\mathbf{n}$ are positive integers, $\mathbf{m} + \mathbf{n}$ and $\mathbf{mn}$ are respectively the sum and the product of $\mathbf{m}$ and $\mathbf{n}$ in the ordinary sense, $\min(\mathbf{m}, \mathbf{n})$ is the lesser of $\mathbf{m}$ and $\mathbf{n}$, and $\mathbf{m} - \mathbf{n}$ is the difference of $\mathbf{m}$ and $\mathbf{n}$ in the ordinary sense if $\mathbf{m} > \mathbf{n}$ and is 1 if $\mathbf{m} \leqq \mathbf{n}$.[3]

Using the preceding definitions, and Theorem II, we obtain the following:

THEOREM IV. *If P conv* 2 *then* $\sim P$ *conv* 1. *If P conv* 1, *then* $\sim P$ *conv* 2. *If P has a normal form different from* 1 *and* 2, *then* $\sim P$ *conv* 3.

THEOREM V. *If* $\mathbf{P}$ *conv* 2 *and* $\mathbf{Q}$ *conv* 2, *then* $\&(\mathbf{P}, \mathbf{Q})$ *conv* 2. *If* $\mathbf{P}$ *conv* 1 *and* $\mathbf{Q}$ *conv* 2, *or if* $\mathbf{P}$ *conv* 2 *and* $\mathbf{Q}$ *conv* 1, *or if* $\mathbf{P}$ *conv* 1 *and* $\mathbf{Q}$ *conv* 1, *then* $\&(\mathbf{P}, \mathbf{Q})$ *conv* 1. *If* $\mathbf{P}$ *and* $\mathbf{Q}$ *have normal forms, and one of the two normal forms is different from* 1 *and* 2, *then* $\&(\mathbf{P}, \mathbf{Q})$ *conv* 3.

In the intuitive interpretation of our formal system, we identify the formula 1 with the truth value *false*, and the formula 2 with the truth value *true*. This identification is artificial, especially since the same two formulas are to be identified with the positive integers *one* and *two*, but it is apparently harmless. The viewpoint taken is that formal logic requires nothing of the ideas *true* and *false* except that they be distinct, which property the formulas 2 and 1 do possess. On this basis, the formulas δ, $\sim$ and $\&$ are identified respectively as equality, negation and logical product.

The theorem that our system is free of contradiction now follows immediately.

THEOREM VI. *There is no formula* $\mathbf{P}$ *such that both* $\mathbf{P}$ *and* $\sim\mathbf{P}$ *are provable.* For suppose that $\mathbf{P}$ is provable. Then, since the process of conversion is reversible, $\mathbf{P}$ conv 2. Therefore, by Theorem IV, $\sim\mathbf{P}$ conv 1. Therefore, by Theorem II, it is not true that $\sim\mathbf{P}$ conv 2. Therefore $\sim\mathbf{P}$ is not provable.

3. *Combinations.*—We define:[4]

$I \rightarrow \lambda x.x.$

$J \rightarrow \lambda fxyz.f(x, f(z, y)).$

$T \rightarrow J(I, I).$

The term *combination* is defined by induction as follows. The formulas I, J, δ and any variable standing alone are combinations. If **M** and **N** are combinations, $\{\mathbf{M}\}(\mathbf{N})$ is a combination.

We define an intuitive operator $\lambda_{\mathbf{x}}|$ such that, if **M** is a combination containing **x** as a free variable, $\lambda_{\mathbf{x}}\mathbf{M}|$ is another combination, which does not contain **x** as a free variable. The definition of this operator is by induction, as follows. $\lambda_{\mathbf{x}}\mathbf{x}|$ is I. If **A** contains **x** as a free variable and **F** does not, $\lambda_{\mathbf{x}}\{\mathbf{F}\}(\mathbf{A})|$ is $J(T, \lambda_{\mathbf{x}}\mathbf{A}|, J(I, \mathbf{F}))$. If **F** contains **x** as a free variable and **A** does not, $\lambda_{\mathbf{x}}\{\mathbf{F}\}(\mathbf{A})|$ is $J(T, \mathbf{A}, \lambda_{\mathbf{x}}\mathbf{F}|)$. If both **F** and **A** contain **x** as a free variable, $\lambda_{\mathbf{x}}\{\mathbf{F}\}(\mathbf{A})|$ is $J(T, T, J(I, J(T, T, J(T, \lambda_{\mathbf{x}}\mathbf{A}|, J(T, \lambda_{\mathbf{x}}\mathbf{F}|, J)))))$.

The *combination belonging* to a formula is defined by induction as follows. The combination belonging to a formula which consists of δ or a variable standing alone is the same as the formula itself. The combination belonging to $\{\mathbf{M}\}(\mathbf{N})$ is $\{\mathbf{M}'\}(\mathbf{N}')$ where $\mathbf{M}'$ and $\mathbf{N}'$ are the combinations belonging to **M** and **N** respectively. The combination belonging to $\lambda\mathbf{x}[\mathbf{R}]$ is $\lambda_{\mathbf{x}}\mathbf{R}'|$, where $\mathbf{R}'$ is the combination belonging to **R**.

Obviously, every well-formed formula has a unique combination belonging to it. And it is readily proved that the combination belonging to a formula is λ-convertible into the formula. Also that the same combination belongs to two formulas **A** and **B** if and only if **A** conv-I **B**. Given a combination, it is constructively possible to determine whether it belongs to a formula and, if so, to obtain the formula.

4. *Metads.*—We introduce the abbreviation, $[\mathbf{a}, \mathbf{b}] \rightarrow \lambda f.f(\mathbf{a}, \mathbf{b})$. And we modify the notion of *metad* employed by Kleene (1935 Part II), defining this term, and the term *rank* of a metad, by induction as follows. The formulas 1, 2, 3, ... (the positive integers) are metads of rank 1, and the formulas obtainable from them by conversion are metads of rank 1. If **a** and **b** are metads, $[\mathbf{a}, \mathbf{b}]$, as well as any formula obtainable from $[\mathbf{a}, \mathbf{b}]$ by conversion, is a metad whose rank is greater by one than the greater of the ranks of the two metads **a** and **b**. The *metad of* a combination is defined by induction as follows. The metad of I is 1, the metad of J is 2, the metad of δ is 3 and the metads of the variables a, b, c, ... are 4, 5, 6, ... respectively. The metad of $\{\mathbf{A}\}(\mathbf{B})$ is $[\mathbf{a}, \mathbf{b}]$, where **a** is the metad of **A** and **b** is the metad of **B**. The *metad belonging* to a formula is defined to be the metad of the combination belonging to the formula.

Using methods of formal definition which are due to Kleene (1935 Part II) and, in the case of thm, the combinatory analysis of λ-conversion

which is due to Rosser (loc. cit.), we are able to define formulas having the following properties.

$\mathfrak{M}_1$ and $\mathfrak{M}_2$ such that, if $[\mathbf{a}, \mathbf{b}]$ is a metad, $\mathfrak{M}_1([\mathbf{a}, \mathbf{b}])$ conv $\mathbf{a}$ and $\mathfrak{M}_2([\mathbf{a}, \mathbf{b}])$ conv $\mathbf{b}$, and if $\mathbf{a}$ is a metad of rank 1, $\mathfrak{M}_1(\mathbf{a})$ conv 1 and $\mathfrak{M}_2(\mathbf{a})$ conv $\mathbf{a}$.

$\mathfrak{G}$ such that, if $\mathbf{a}$ is a metad belonging to a formula $\mathbf{A}$ which contains no free variables, $\mathfrak{G}(\mathbf{a})$ conv $\mathbf{A}$.

met such that, if $\mathbf{A}$ is a formula which contains no free variables and has a normal form, met($\mathbf{A}$) is convertible into the metad belonging to the normal form of $\mathbf{A}$.

$\mathfrak{n}$ such that, if $\mathbf{a}$ is the metad belonging to one of the formulas $\lambda fx.f(x)$, $\lambda fx.f(f(x))$, $\lambda fx.f(f(f(x)))$, ..., then $\mathfrak{n}(\mathbf{a})$ conv 2, and if $\mathbf{a}$ is any other metad, then $\mathfrak{n}(\mathbf{a})$ conv 1.

thm, a formula which enumerates the metads belonging to provable formulas, in the sense that in the infinite sequence, thm(1), thm(2), thm(3), ..., every term is a metad belonging to a provable formula, and every metad which belongs to a provable formula occurs at least once.

5. *Quantifiers.* We define:

$N \rightarrow \lambda x.\mathfrak{n}(\mathrm{met}(x))$.

$\iota \rightarrow \lambda f\mathfrak{G}(\mathfrak{M}_2(\mathrm{thm}(\mathfrak{p}(\lambda n.\ \delta(f, \mathfrak{G}(\mathfrak{M}_1(\mathrm{thm}(n)))), 1)))))$.

$\Sigma \rightarrow \lambda f.f(\iota(f))$.

$\Xi \rightarrow \lambda fgx.\ \delta(f, \mathfrak{G}(\mathfrak{M}_1(\mathrm{thm}(\mathfrak{p}(\lambda n.\sim[\sim\delta(f, \mathfrak{G}(\mathfrak{M}_1(\mathrm{thm}(n)))).$
$\sim\delta(g, \mathfrak{G}(\mathfrak{M}_1(\mathrm{thm}(n))))].\delta(x,\mathfrak{G}(\mathfrak{M}_2(\mathrm{thm}(n)))),\ 1)))))$.

Here $\mathfrak{p}$ is the Kleene $\mathfrak{p}$-function (defined in Kleene 1935 Part II), which has the property that, if, for every positive integer $\mathbf{m}$, $\mathbf{F}(\mathbf{m})$ conv 1 or 2, and $\mathbf{r}$ is the least positive integer $\mathbf{m}$, $\geqq \mathbf{n}$, such that $\mathbf{F}(\mathbf{m})$ conv 2, then $\mathfrak{p}(\mathbf{F}, \mathbf{n})$ conv $\mathbf{r}$.

The formula $\mathbf{N}$ has the property that, if $\mathbf{X}$ has a normal form, $\mathbf{N}(\mathbf{X})$ conv 2 or 1 according to whether $\mathbf{X}$ is or is not a positive integer. The formula ι has the property that, if there is any $\mathbf{X}$ such that $\mathbf{F}(\mathbf{X})$ conv 2, then $\iota(\mathbf{F})$ is such an $\mathbf{X}$, and in the contrary case $\iota(\mathbf{F})$ has no normal form. Consequently $\Sigma(\mathbf{F})$ conv 2 if there is any $\mathbf{X}$ such that $\mathbf{F}(\mathbf{X})$ conv 2, and in the contrary case $\Sigma(\mathbf{F})$ has no normal form. The formula Ξ has the property that, if $\mathbf{F}(\mathbf{X})$ conv 2 and $\mathbf{G}$ has a normal form but $\mathbf{G}(\mathbf{X})$ is not convertible into 2 then $\Xi(\mathbf{F}, \mathbf{G}, \mathbf{X})$ conv 2, and if $\mathbf{G}(\mathbf{X})$ conv 2 and $\mathbf{F}$ has a normal form but $\mathbf{F}(\mathbf{X})$ is not convertible into 2 then $\Xi(\mathbf{F}, \mathbf{G}, \mathbf{X})$ conv 1, and if neither $\mathbf{F}(\mathbf{X})$ nor $\mathbf{G}(\mathbf{X})$ is convertible into 2 then $\Xi(\mathbf{F}, \mathbf{G}, \mathbf{X})$ has no normal form.

Making use of metads and of the $\mathfrak{p}$-function, it is possible to define a set of formulas Π_1, Π_2, Π_3, ..., Π_ω, $\Pi_{\omega+1}$, ..., related to the universal quantifier as follows. If $\mathbf{G}(\mathbf{x})$ is found among the consequences of $\mathbf{F}(\mathbf{x})$

where $\mathbf{x}$ is a variable, then $\Pi_\alpha(\mathbf{F}, \mathbf{G})$ conv 2, and in any other case $\Pi_\alpha(\mathbf{F}, \mathbf{G})$ has no normal form.

For this purpose, of course, the notion of the consequences of a formula must be defined formally. It turns out, as might be expected in connection with the theorem of Gödel,[5] that there is no all inclusive definition of the notion of the consequences of a formula, but merely a sequence, extending into the second number class, of more and more inclusive definitions of the notion. The subscript α, in the symbol Π_α, refers to the different definitions of the notion of the consequences of a formula.

If α is of the second kind, the definition of Π_α has a special form, as may be illustrated in connection with the definition of Π_ω. It is possible to define a formula π such that, for every positive integer $\mathbf{n}$, $\pi(\mathbf{n})$ conv Π_n. Then, $\Pi_\omega \rightarrow \lambda fg\Sigma x . N(x) . \pi(x, f, g)$.

We also define:

$$\Pi^\alpha \rightarrow \Xi(\Pi_\alpha(N), \lambda f\Sigma x . N(x) . \sim f(x)).$$
$$\Sigma^\alpha \rightarrow \lambda f \sim . \Pi^\alpha x . \sim f(x).$$

If now we select out of the second number class a particular constructively defined ordinal α of the second kind, and identify Π^α and Σ^α respectively with the intuitive notions of the universal quantifier over the class of positive integers and the existential quantifier over the class of positive integers, it is believed that the system of logic with which we are dealing will then be found to be adequate to a portion of elementary number theory which can be described as the portion which does not require proofs by induction of higher than a certain order. In particular, if α is ω, proofs by induction of any finite order are possible. And it is believed that if α increases indefinitely, the order of induction possible increases indefinitely, so that we obtain, in a certain sense, a metamathematical proof of the freedom from contradiction of elementary number theory.

Of course, the considerations adduced by Gödel (loc. cit.) make it certain that the intuitive proof that Π^α has the properties of a universal quantifier cannot be formalized in terms of Π^α as the universal quantifier (although the intuitive proof probably can be formalized in terms of a quantifier of higher order). The proof of freedom from contradiction which we have outlined is believed, nevertheless, to have an interest on the ground that it is easier to recognize a particular intuitive proof as valid than it is to recognize intuitively the general proposition that every proof formalizable in a given system is valid.

There is also, perhaps, the possibility that the methods here used might lead to a proof of the consistency of some portion of analysis.

[1] Alonzo Church, *Ann. Math.*, 2nd ser., **33**, 346–366 (1932), and **34**, 839–864 (1933). S. C. Kleene, *Ann. Math.*, 2nd ser., **35**, 529–544 (1934). We shall refer to the latter paper as Kleene 1934.

[2] Alonzo Church and J. B. Rosser, "Some Properties of Conversion."

[3] Explicit formal definitions are given by S. C. Kleene, *Amer. Jour. Math.*, **57**, 153–173 (1935). This paper, and a second part of it, forthcoming in the *Amer. Jour. Math.*, will be referred to as Kleene 1935.

[4] Cf. J. B. Rosser, *Ann. Math.*, 2nd ser., **36**, 127–150 (1935), and a second part of the same paper, forthcoming in the *Ann. Math.* Cf. also Kleene, 1934, p. 536.

[5] Kurt Gödel, *Monatsh. fur Math. u. Phys.*, **38**, 173–198 (1931).

INTENSITIES OF VIBRATION ROTATION BANDS

BY JENNY E. ROSENTHAL

CHEMISTRY DEPARTMENT, COLUMBIA UNIVERSITY

Communicated February 18, 1935

The intensities of some of the HCl bands have been calculated by Dunham;[1,2] his results (obtained by means of a rather laborious mathematical method) involve, however, certain approximations and are limited to the fundamental band and the first few harmonics. The *exact* expression for the probability of any transition $v' \leftarrow v''$ is derived here by a simple mathematical method for the case of a Morse potential function between the nuclei. The standard band spectroscopic notation is used throughout, and only additional symbols are defined explicitly.

Let $P(r)$ be the electric dipole moment considered as a function of the internuclear distance. The $(v'v'')$ element of the matrix for P, which determines the transition probability $v' \leftarrow v''$ is:[3]

$$P_{v'v''} = \int_0^\infty \varphi_{v'}^*(r) P(r) \varphi_{v''}(r) dr = \int_{-\infty}^{+\infty} \varphi_{v'}^* P \varphi_{v''} dr; \qquad (1)$$

φ_v is the wave function of an oscillator obeying the Morse potential function and is known to have the following form:[4,5,6]

$$\varphi_v = A_v a^{1/2} e^{-z/2} \sum_{s=0}^{v} \frac{(-1)^s z^{k/2-s-1/2}}{s!(v-s)!\Gamma(k-v-s)}, \qquad (2)$$

where:

$k = 1/x_e$, $z = ke^{-a(r-r_e)}$, $ar_e = (\omega_e x_e/B_e)^{1/2}$ and

$$A_v = [v!(k-2v-1)\,\Gamma(k-v)]^{1/2}$$

We introduce $\xi = (r-r_e)/r_e$ and expand $P(\xi)$ around $\xi = 0$;

$$P(\xi) = P_e + P_e'\xi + P_e''\xi^2/2! + \ldots + P_e^{(n)}\xi^n/n!, \qquad (3)$$

where $P_e^{(n)} = (d^n P/d\xi^n)_{\xi=0}$.

It is usually assumed that in this expansion all the terms after the second one may be neglected for small values of ξ. Dunham, however, has made approximate calculations where he took into account higher order terms. We shall derive $P_{v'v''}$ first for the case $P_e^{(n)} = 0$ if $n > 1$, then the effect of $P_e'' \neq 0$ will be calculated for the $v' \leftarrow 0$ transitions. If we take into account orthogonality relations, then:

$$P_{v'v''} = r_e P_e' \int_{-\infty}^{+\infty} \xi \varphi_{v'} \varphi_{v''} d\xi = -(P_e'/a^2 r_e) \int_0^{\infty} \varphi_{v'}(z) \varphi_{v''}(z) \log z \, dz/z. \quad (4)$$

Up to this point this derivation, while given in a different form, agrees with Dunham's. A different mathematical procedure is now introduced. If we substitute for φ_v its value given by Eq. (2), Eq. (4) becomes:

$$P_{v'v''} = -(P_e'/ar_e) A_{v'} A_{v''} \times \int_0^{\infty} e^{-z} \sum_{s=0}^{v'} \sum_{t=0}^{v''} \frac{(-1)^{s+t} z^{k-2-s-t} \log z dz}{s!(v'-s)!\,\Gamma(k-v'-s) t!(v''-t)!\,\Gamma(k-v''-t)} \quad (5)$$

After performing the integration,[7] we obtain:

$$P_{v'v''} = -(P_e'/ar_e) A_{v'} A_{v''} \sum_{t=0}^{v''} [(-1)^t t!(v''-t)!\,\Gamma(k-v''-t)]^{-1} \sum_{s=0}^{v'} C_s^{(t)} \psi(k-s-t-1) \quad (6)$$

where

$$C_s^{(t)} = (-1)^s \Gamma(k-s-t-1)/[s!(v'-s)!\,\Gamma(k-v'-s)]$$

and, as usual,

$$\psi(y) = d \log \Gamma(y)/dy.$$

Eq. (6) may be greatly simplified by the following method: since

$$\psi(y+1) - \psi(y) = 1/y, \quad (7)$$

$$\sum_{s=0}^{v'} C_s^{(t)} \psi(k-s-t-1) \equiv \sum_{s=0}^{v'-1} (k-s-t-2)^{-1} \sum_{j=0}^{s} C_j^{(t)} + \psi(k-t-v'-1) \sum_{s=0}^{v'} C_s^{(t)}. \quad (8)$$

We shall now show that:

$$\sum_{s=0}^{v'} C_s^{(t)} \equiv 0. \quad (9)$$

If $v'' = 0$, then:

$$\sum_{s=0}^{v'} C_s^{(0)} = [\Gamma(k)/A_{v'} A_{v''}] \int_0^{\infty} \varphi_{v'} \varphi_0 dz/z = 0, \quad (10)$$

due to the orthogonality of the functions φ. For the same reason,

$$\sum_{t=0}^{v''} [(-1)^t(v''-t)!t!\,\Gamma(k-v''-t)]^{-1} \sum_{s=0}^{v'} C_s^{(t)} = 0. \qquad (11)$$

The validity of expression (9) follows immediately by recurrence from Eqs. (11) and (10). Hence:

$$P_{v'v''} = -(P_e'/ar_e)A_{v'}A_{v''} \sum_{t=0}^{v''} \sum_{s=0}^{v'-1} \sum_{j=0}^{s} C_j^{(t)}[(-1)^t(v''-t)!t!\,\Gamma(k - v''-t)(k-s-t-2)]^{-1}. \qquad (12)$$

The summations may be performed quite easily, particularly for low values of v' and v''. The expanded formulae for $P_{v'v''}$ are given in table 1; the absolute values agree roughly with Dunham's, but, apparently, in his calculations there is a mistake in sign.

We may now consider the correction term in the transition probability due to $P_e'' \neq 0$. Let $\overline{P}_{v'0} = P_{v'0} + S_{v'0}P_e''$. This correction term $S_{v'0}$ may be calculated by means analogous to those used in the determination of $P_{v'v''}$. The more important steps in this derivation are given concisely below.

$$S_{v'0} = r_e \int_{-\infty}^{+\infty} \frac{1}{2}\,\xi^2 \varphi_{v'}\varphi_{v''} d\xi = \frac{1}{2}(a^3 r_e^2)^{-1} \int_0^{\infty} \varphi_{v'}\varphi_0 (\log z - \log k)^2 dz/z \qquad (13)$$

$$S_{v'0} = (A_{v'}A_0/a^2r_e^2) \sum_{s=0}^{v'} \frac{(-1)^s\Gamma(k-s-1)}{s!(v'-s)!2\Gamma(k)\Gamma(k-v'-s)} \left[\frac{d\psi(k-s-1)}{d(k-s-1)} + \psi^2(k-s-1) - 2\log k\,\psi(k-s-1)\right]. \qquad (14)$$

With the use of Eq. (7), we obtain after some rearrangements:

$$S_{10} = \frac{-A_1A_0}{(k-2)a^2r_e^2\Gamma(k)} \left[\log k - \psi(k) + \frac{1}{(k-1)} + \frac{1}{(k-2)}\right]. \qquad (15)$$

For $v' \geqslant 2$

$$S_{v'0} = \frac{A_{v'}A_0}{a^2r_e^2\Gamma(k)} \left\{ \sum_{s=0}^{v'} \frac{1}{k-s-2} \left[\psi(k) - \log k - \frac{1}{k-s-1} - \frac{1}{k-s-2}\right] \sum_{j=0}^{s} \frac{(-1)^j\Gamma(k-j-1)}{j!(v'-j)!\Gamma(k-v'-j)} - \sum_{s=1}^{v'-1} \frac{1}{k-s} \sum_{m=s}^{v'-1} \frac{1}{k-m-2} \sum_{j=0}^{m} \frac{(-1)^j\Gamma(k-j-1)}{j!(v'-j)!\Gamma(k-v'-j)} \right\} \qquad (16)$$

It is known[8] that

$$\psi(k) = \log k - \frac{1}{2k} + \sum_{r=1}^{\infty} \frac{(-1)^r B_r}{2r} \left(\frac{1}{k}\right)^{2r} \qquad (17)$$

TABLE 1

EXPLICIT EXPRESSIONS FOR $P_{v'v''}/P_e'$

v''/v'	1	2	3	4
0	$-\frac{1}{1-2x_e}\left[\frac{B_e(1-3x_e)}{\omega_e}\right]^{1/3}$	$-\frac{1}{1-3x_e}\left[\frac{B_ex_e(1-5x_e)}{2\omega_e(1-2x_e)}\right]^{1/3}$	$-\frac{1}{1-4x_e}\left[\frac{2B_e(1-7x_e)}{3\omega_e(1-2x_e)(1-3x_e)}\right]^{1/3}$	$-\frac{1}{1-5x_e}\left[\frac{3B_ex_e(1-9x_e)}{2\omega_e(1-2x_e)(1-3x_e)(1-4x_e)}\right]^{1/3}$
1		$-\frac{1}{1-4x_e}\left[\frac{2B_e(1-3x_e)(1-5x_e)}{\omega_e(1-2x_e)}\right]^{1/3}$		

The expression $[\ ]^{1/3}$ corresponds always to the positive root.

where the B_r represent Bernoulli's numbers. For large values of k, the summation with respect to r is a very small number which converges rapidly. Expressions (15) and (16) may thus be evaluated to any desired degree of accuracy.

The transition probabilities $P_{v'v''}$ may be correlated to the observed intensities of the bands. For two transitions $1 \leftarrow 0$ and $v' \leftarrow 0$, the ratio of the integrated absorption coefficients is given by:

$$\alpha_{v'0}/\alpha_{10} = (\omega_{v'0}/\omega_{10})(P_{v'0}/P_{10})^2 \qquad (18)$$

Explicit expressions for this ratio together with the corresponding numerical values for HCl, HBr and HF are given in table 2. The calculations were made using the most recent (and probably most accurate) values for the vibrational constants.[9,10,11] The figures for HF, where the vibrational energy may not be accurately represented by the Morse formula, are only approximate. For this molecule, it was assumed that $x_e = 0.0205$, a value which seems to give the best results in correlating the observed ω_{10}, ω_{30} and ω_{40} with the Morse formula.

Comparison with experimental data is possible only for the HCl molecule, where the observed value for α_{20}/α_{10} (as determined by Dunham[1] and Bourgin[12]) is 1.61×10^{-2}. Bartholomé[13] has raised objections against Bourgin's method, but, according to Kemble[14] the experimental result given above is essentially correct. The value 1.61×10^{-2} agrees with the theoretical calculation well within the limits of experimental error. In spite of the close agreement between experiment and the theory derived for the case $P_e'' = 0$, Dunham[2] came to the conclusion that on the basis of the experimental evidence we cannot decide whether $P_e'' \sim 0$ or $P_e''/P_e' \sim 4.3$. In the form in which Dunham gives his expression for α_{20}/α_{10}, the meaning of the

second solution for P_e'' is not clear. We may find it, however, in the following way. If we define

$$(\bar{\alpha}_{20}/\bar{\alpha}_{10}) \equiv (\omega_{20}/\omega_{10})(\bar{P}_{20}/\bar{P}_{10})^2 \equiv (\omega_{20}/\omega_{10})\left[\frac{P_{20} + P_e''S_{20}}{P_{10} + P_e''S_{10}}\right]^2 \qquad (18')$$

we obtain from the experimental fact $\bar{\alpha}_{20}/\bar{\alpha}_{10} \sim \alpha_{20}/\alpha_{10}$ the relationship

$$\left[\frac{P_{20} + P_e''S_{20}}{P_{10} + P_e''S_{10}}\right]^2 \sim \left[\frac{P_{20}}{P_{10}}\right]^2 \qquad (19)$$

Since $S_{20} \neq S_{10}$, Eq. (19) has two possible solutions; one of these corresponds, however, to $\bar{P}_{20}/\bar{P}_{10} \sim -P_{20}/P_{10}$. A value of P_e'' which is large enough to make one of the matrix components for P change its sign indicates that the expansion (3) diverges. In this case, the whole treatment would have to be discarded, since it is meaningless to approximate a diverging expansion by its first two terms. This second value of P_e'' can thus be eliminated as corresponding to an extraneous mathematical solution. The case: $\bar{P}_{20}/\bar{P}_{10} \sim P_{20}/P_{10}$ leads to $P_e'' \sim 0$. We may thus conclude that in the neighborhood of the equilibrium position the electric moment of HCl is very nearly linear.

TABLE 2

EXPLICIT EXPRESSION FOR THE RATIO OF THE ABSORPTION COEFFICIENTS $\alpha_{v'0}/\alpha_{10}$ AND NUMERICAL VALUES FOR THIS RATIO FOR HCl, HBr AND HF

v'	2	3	4
	$\frac{x_e(1-5x_e)}{(1-3x_e)^2}$	$\frac{2x_e^2(1-7x_e)}{(1-3x_e)^2(1-4x_e)}$	$\frac{6x_e^3(1-9x_e)}{(1-3x_e)^2(1-4x_e)(1-5x_e)}$
HCl	1.7576×10^{-2}	0.628×10^{-3}	0.343×10^{-4}
HBr	1.734×10^{-2}	0.612×10^{-3}	0.33×10^{-4}
HF	2.09×10^{-2}	0.89×10^{-3}	0.58×10^{-4}

[1] J. L. Dunham, *Phys. Rev.*, **34**, 438 (1929).

[2] J. L. Dunham, *Ibid.*, **35**, 1347 (1930).

[3] The Morse eigenfunctions are normalized for values of r ranging from $-\infty$ to $+\infty$; and it is considered that the value of any definite integral involving these functions is not altered perceptibly by changing the lower limit of integration from $r = 0$ to $r = -\infty$.

[4] P. M. Morse, *Phys. Rev.*, **34**, 57 (1929).

[5] E. U. Condon and P. M. Morse, *Quantum Mechanics*, p. 72.

[6] In both ref. 4 and ref. 5, an extraneous factor is included in the constant A_v, but the values of the eigenfunctions as given by Dunham from Morse's manuscript are correct

[7] Whittaker and Watson, *Modern Analysis*, p. 243.

[8] Ref. 7, p. 252.

[9] J. D. Hardy, E. F. Barker and D. M. Dennison, *Phys. Rev.*, **42**, 279 (1932).

[10] E. K. Plyler and E. F. Barker, *Ibid.*, **44**, 984 (1933).

[11] E. O. Salant and D. E. Kirkpatrick, *Ibid.*, **46**, 315 (1934).

[12] D. C. Bourgin, *Ibid.*, **29**, 794 (1927).

[13] E. Bartholomé, *Zeit. Phys. Chem.*, **B23**, 131 (1933).

[14] E. C. Kemble, *Jour. Chem. Phys.* (May, 1935).

WHAT ARE THE RESTING AND THE ACTIVE STATES OF CHROMATOPHORES?

By G. H. Parker

Biological Laboratories, Harvard University

Read before the Academy, Monday, April 22, 1935

In his classic monograph on the color changes of the chameleon Brücke (1852, p. 28) makes the statements that the light tint of this lizard with its contracted elements corresponds to an active state and that the dark tint, the result of nerve cutting as well as of environmental conditions, represents a passive state of the color cells. This is the view that has been accepted for a long time by most workers in this field including Keller (1895), von Frisch (1912), Giersberg (1930) and others. Von Frisch (1912, p. 174) in particular has pointed out that when the chromatophoral nerves in a fish are stimulated electrically the melanophores contract and that when these nerves are cut the melanophores relax and expand. Such reactions are so closely parallel to what is known in ordinary nerve-muscle preparations that it is only natural to interpret them from this standpoint and to designate the condition of a melanophore with fully concentrated pigment as that of activity and with fully dispersed pigment as that of rest. Carlton (1903, p. 272), however, in his study of the color changes in the lizard Anolis found ground for the reverse assumption in that in this animal the condition of concentrated pigment was believed by him to be one not of activity but of rest. Babák (1913, p. 469), whose work was on amphibians, took still another view and maintained that the two extreme conditions, that of maximum concentration of pigment and that of full dispersion, were each a state of activity as contrasted with a resting presumably intermediate phase. The present paper, which is based upon an experimental study of the states of rest and of activity in the melanophores of the fish Fundulus heteroclitus, is intended to throw light on this somewhat vexed question.

The problem has been attacked from the standpoint of the caudal band in Fundulus which can be produced by cutting near the base of the fish's tail one or more bundles of radiating nerve-fibres. Half a minute or so after this has been done in a live, light-tinted fish, a dark band begins to appear in that the melanophores of the denervated area thus produced disperse their pigment and give rise to a dark streak which, beginning at the cut, extends to the posterior edge of the tail. The state of the melanophores in this streak has been regarded by most workers as the result of nerve paralysis, and the melanophores themselves have been described as in a condition of rest. That such is really not the case has already been shown (Parker, 1934c) for areas of this kind are far from

inert and under appropriate stimuli are capable of a great variety of responses quite inconsistent with a state of paralysis.

When such a dark caudal band as that just described is allowed to remain in a light-tinted fish it will in the course of a day or two blanch and become almost indistinguishable from the rest of the animal's tail. This blanching is not the result of nerve degeneration, for that is known not to take place till some days later (Parker and Porter, 1933). Moreover, if, after the band-like area has blanched, a new cut is made distal to the old one, the dark band will revive showing that the nerve-fibres first cut are still active and open to renewed stimulation. They are in no sense in a state of paralysis, for these fibres can readily reëxcite the dispersion of the melanophore pigment. Not only are there fibres concerned with the dispersion of pigment, but, as was clearly pointed out by von Frisch (1910), there are fibres that will induce its concentrations. If the medulla of a normal Fundulus is stimulated electrically, the fish, including the whole of its tail, will become light in less than a minute. This reaction is not shown, however, by a dark caudal band should one be present. It is difficult to interpret this response in any other way than that suggested by von Frisch, namely, that the concentration of melanophore pigment is excited through nerves. Thus from tests with caudal bands and with electrical stimulation there appears to be good evidence not only that nerves act as agents for the control of melanophores but that there are two sets of such nerves. This view, originally, expressed by Bert (1875), and since supported by a number of investigators, in recent years especially by Smith (1931), has received unusual confirmation in two sets of experimental tests on Fundulus. If the edge of a caudal band in this fish is closely watched melanophore for melanophore while the fish is made to change from light to dark and the reverse, it will be found that the boundaries of the band do not remain constant (Mills, 1932). Some melanophores near the edge of the band can concentrate their pigment but not fully disperse it while others can disperse it but fail to concentrate it. This is precisely what would be expected from cutting in a gross way bundles of nerve fibres some of which were dispersing and some concentrating in their action. On the margins of the band in such a preparation melanophores might well occur in which only one of the two sets of nerve fibres had been eliminated and under such circumstances incomplete melanophore reaction would be the result. Single innervation would never lead to such a condition.

Further confirmatory evidence for double innervation has been found in the regeneration of chromatophoral nerves (Abramowitz, 1935, p. 140). If the regeneration of nerve-fibres in the tail of Fundulus is followed closely, it will be found that when the new growth of fibres invades the old band as indicated by the ability of the melanophores in that band to change

from light to dark or the reverse, the front edge of the wave of light regeneration does not always agree with the front edge of the dark wave and that consequently two sets of fibres with separate functions must be assumed. Thus the detailed study of caudal bands during color changes and during nerve regeneration makes it reasonably certain that the melanophores of these bands in Fundulus possess a double innervation.

Although the double innervation of the melanophores of Fundulus seems well established it is not to be supposed that all chromatophoral systems are necessarily thus constituted. In crustaceans, for instance, notwithstanding the declarations of Pouchet and others to the contrary, nerves appear to play no direct part in the color changes. These changes are controlled completely by blood-borne neurohumors (Brown, 1933). In the dogfish the dark phase has been shown to be due to a blood-soluble pituitary secretion (Lundstrom and Bard, 1932) and the light phase to nerves which produce a neurohumor probably soluble in lipoids (Parker and Porter, 1934; Parker, 1934b). These two instances together with that of Fundulus show that chromatophoral systems in different animals may be very differently organized so far as nerves are concerned, no direct innervation, single innervation and double innervation having been already identified. In all such instances, however, both phases in each case appear to be subject to neurohumoral control.

Having made clear the more recent views on the activities of color cells, particularly melanophores, the problem of their resting and their active states may now be taken up. It must be apparent from what has been said about the melanophores of Fundulus that the older idea that the state of concentrated pigment represents activity and that of dispersed pigment rest can no longer be maintained. The original analogy with the nerve-muscle preparation is no longer possible for the whole scheme of innervation as already elucidated is quite at variance with this comparison. How then can we best approach the subject experimentally to determine whether there are or are not conditions of rest and of activity in these color cells?

From the account that has been given of the melanophores of Fundulus it might be assumed that a resting state for these cells could be attained if they were completely freed from the influence of their two sets of nerves with their neurohumors. This state might possibly be reached after the degeneration of these nerves. In an attempt to attain this condition some thirty Fundulus were subjected to caudal-band operations and were set aside in large white-walled illuminated aquaria for the degeneration of their nerves. After eleven days four of these fishes were tested to ascertain if degeneration had been fully completed. This test consisted in making a new cut in the tail just distal to the old one by which the original band had been formed and in watching for a recurrence

of the band. In no instance did the band reappear and it was therefore assumed that the nerves had completely and fully degenerated. In the four fishes tested and in ten others, the remainder of the original thirty, the caudal bands had completely disappeared. Their positions could be determined through the scars of the cuts by which they had been produced, but their melanophores, though in the beginning in the dark phase, were now only slightly stellate and agreed in this respect with those of the tail in general.

Thus it might be assumed that the stellate condition represents the

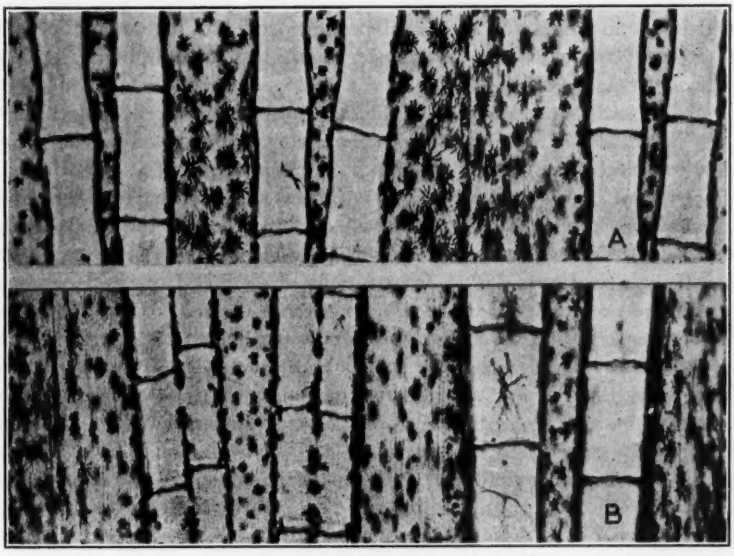

FIGURE 1

Magnified surface views across two caudal bands from Fundulus: *A*, a dark band as shown by the melanophores with dispersed pigment (central part of figure) from a dark fish which has been long enough in a light-walled vessel to have rendered it light as seen in the lateral melanophores with concentrated pigment; *B*, a light band (central part of figure) from a light fish rendered dark (lateral melanophores) by black surroundings.

condition of rest. But in this instance the stellate condition is also characteristic of the melanophores in the tail as a whole and it has already been shown that an adjacent innervated area in the tail will in a day or so induce in a band through its neurohumors the same melanophore state that the area itself has (Parker, 1934a). The condition of the melanophores in the band then may not be one of rest but one merely due to the influence of the adjacent area.

To test this view a second set of Fundulus was prepared in the same way as the first except that the new fishes were kept continuously in a black-walled illuminated vessel instead of in a white-walled one. After a little less than two weeks they were carefully inspected. In thirteen fishes that remained of the original twenty the caudal bands had entirely disappeared in that their areas were occupied by melanophores that had nearly fully dispersed pigment and that in this respect were the same as those in the rest of the tail. Here as in the instance of the light Fundulus the band assumed the tint of the tail in general, a condition that favors the idea that the fully denervated melanophores are not in a resting state but in one in which they assume the condition of the surrounding melanophores probably through the transfer to them of neurohumors from the adjacent fields (Parker, 1934a).

To test this question still further a few fishes from each of the two sets just described were placed in aquaria contrary to those in which they had been formerly; that is, Fundulus that were dark in tint were put in a light-walled illuminated vessel and vice versa. In a quarter of an hour the light fishes were dark and the dark ones light except the bands on the tail. In the newly darkened fishes these bands remained light and in the newly lightened ones dark. Preparations of one fish from each set were made and are shown in figure 1. The remaining fishes were allowed to rest in their new aquaria and after about two days they were again inspected. In all instances the bands had practically disappeared thus confirming the idea that a denervated caudal band is in the end under the domination, so far as its melanophores are concerned, of the adjacent parts of the tail.

As a final test a dark fish with an indistinguishable dark band was injected with 0.1 cc. of adrenalin, one part in a thousand of water, and a light fish with a correspondingly light band was etherized. Both changed completely, band and all, the dark fish light and the light fish dark, showing that the completely denervated melanophores were still fully active and in no sense unresponsive.

From these several tests I conclude that the active phase of a melanophore as contrasted with a state of rest is not necessarily to be found in the state with concentrated pigment nor in that with dispersed pigment nor at some intermediate point. Determinations of this kind, based in general on a comparison with nerve-muscle preparations seem to me to be beside the point. In my opinion any state of pigment immobility in a melanophore is a resting state irrespective of whether that state is one of full dispersion or of full concentration or some intermediate state. The active condition is one in which the pigment particles are moving either inward or outward. It is interesting to note that from this standpoint as can be seen by the inspection of living melanophores in the isolated scales of

Fundulus, the active phase is one in which the Brownian movement of the melanin particles is very obvious and the resting phase one in which this type of movement is greatly reduced. This state of affairs suggests that the active condition of a melanophore is one in which its protoplasm is mostly in the sol state and the resting phase is one in which the gel state predominates. So far as resting and active states of chromatophores are concerned the conclusions from the present study conform almost exactly with those expressed by Wyman in 1924. They further support the opinion that the momentary state of any chromatophore is in the main determined by its immediate neurohumoral environment. I am inclined to accept this view notwithstanding the fact that in almost all animals with chromatophoral functions the loss of the eyes is followed by a permanent moderate darkening of the skin (Parker, Brown and Odiorne, 1935), a condition which according to the generalization just advanced would call for some measure of balance in neurohumoral activity.

Summary.—1. In melanophores neither the state of concentrated pigment nor of dispersed pigment nor any intermediate condition is necessarily a state of rest. Such conditions have been interpreted for the most part from the standpoint of nerve-muscle preparations, an inappropriate comparison.

2. If the terms rest and activity are to be used for chromatophores, rest is a state of quiescence of the melanin particles and activity of motion of these particles.

3. Such a state of activity can be shown to be associated with much Brownian movement among the melanin grains and the state of rest with very little of this movement suggesting that in the state of activity the melanophore protoplasm has the character of a sol and in that of rest the character of a gel.

Abramowitz, A. A., "Regeneration of Chromatophore Nerves," *Proc. Nat. Acad. Sci.*, **21**, 137–141 (1935).

Babák, E., "Ueber den Einfluss des Lichtes auf die Vermehrung der Hautchromatophoren," *Arch. ges. Physiol.*, **149**, 462–470 (1913).

Bert, P., "Sur le méchanisme et les causes des changements de couleur chez le Caméléon," *C. R. Acad. Sci. Paris*, **81**, 938–941 (1875).

Brown, F. A., Jr., "The Controlling Mechanism of Chromatophores in Palaemonetes," *Proc. Nat. Acad. Sci.*, **19**, 327–329 (1933).

Brücke, E., "Untersuchungen über den Farbenwechsel des Afrikanischen Chamäleons," *Denkschr. Akad. Wiss. Wien, math.-naturwiss. Cl.*, **4**, 179–210 (1852).

Carlton, F. C., "The Color Changes in the Skin of the So-Called Florida Chameleon, Anolis carolinensis Cuv.," *Proc. Amer. Acad. Arts Sci.*, **39**, 259–276 (1903).

Von Frisch, K., "Ueber die Beziehungen der Pigmentzellen in der Fischhaut zum sypmathischen Nervensystem," *Festschr. R. Hertwig*, **3**, 15–28 (1910).

Von Frisch, K., "Ueber farbige Anpassung bei Fischen," *Zool. Jahrb. Abt. allg. Zool. Physiol.*, **32**, 171–230 (1912).

Giersberg, H., "Der Farbwechsel der Fische," *Zeit. vergl. Physiol.*, **13**, 258–279 (1930).

Keller, R., "Ueber den Farbenwechsel des Chamaeleons und einiger anderer Reptilien," *Arch. ges. Physiol.*, **61**, 123–168 (1895).

Lundstrom, H. M., and Bard, P., "Hypophysial Control of Cutaneous Pigmentation in an Elasmobranch Fish," *Biol. Bull.*, **62**, 1–9 (1932).

Mills, S. M., "The Double Innervation of Fish Melanophores," *Jour. Exp. Zoöl.*, **64**, 231–244 (1932).

Parker, G. H., "Cellular Transfer of Substances, Especially Neurohumors," *Jour. Exper. Biol.*, **11**, 81–88 (1934a).

Parker, G. H., "Oil-Soluble and Water-Soluble Neurohumors," *Anat. Rec.*, **60**, Suppl. 30 (1934b).

Parker, G. H., "The Prolonged Activity of Momentarily Stimulated Nerves," *Proc. Nat. Acad. Sci.*, **20**, 306–310 (1934c).

Parker, G. H., Brown, F. A., Jr., and Odiorne, J. M., "The Relation of the Eyes to Chromatophoral Activities," *Proc. Amer. Acad. Arts Sci.*, **69**, 439–462 (1935).

Parker, G. H., and Porter, H., "Regeneration of Chromatophore Nerves," *Jour. Exper. Zoöl.*, **66**, 303–309 (1933).

Parker, G. H., and Porter, H., "The Control of the Dermal Melanophores in Elasmobranch Fishes," *Biol. Bull.*, **66**, 30–37 (1934).

Smith, D. C., "The Action of Certain Autonomic Drugs upon the Pigmentary Responses of Fundulus," *Jour. Exp. Zoöl.*, **58**, 423–453 (1931).

Wyman, L. C., "Blood and Nerve as Controlling Agents in the Movements of Melanophores," *Jour. Exper. Zoöl.*, **39**, 73–131 (1924).

THE GROWTH HORMONE AND THE DWARF TYPE OF GROWTH IN CORN

BY J. VAN OVERBEEK

WILLIAM G. KERCKHOFF LABORATORIES OF THE BIOLOGICAL SCIENCES, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated April 6, 1935

1. *Introduction.*—In this paper is given a short report of an analysis of the causes of one of the dwarf types of growth in corn. A more detailed report will be published later. The dwarf type investigated is *nana*,[1] which differs from the normal race by one single gene. The seeds were obtained from Dr. E. G. Anderson, of the California Institute, and were the result of a cross between F_1 (*naNA*) with *na na*. As *na* is recessive the plants from these seeds were 50% dwarfs and 50% normals. The difference between nana and normal consists in an inhibition of the growth in length in nana. Even in seedlings, especially when grown in the dark, nana can be distinguished from normal by its shorter length. This difference in growth is caused by a growth inhibition of the mesocotyl of nana. *The length of the coleoptile is the same in both races* (Fig. 1).

The seeds were soaked for several hours, then set out in moist sand and grown in a dark room at constant temperature (24°C.) and constant

humidity (90%). The dark room was illuminated with orange light. A Corning filter No. 348 was used, light from which produced no phototropic curvature in corn or in Avena. Four to six days after the seeds were set out, the plants were ready to be used for the experiments. The primary leaf had not then broken through; the length of coleoptile and mesocotyl can be seen in figure 1.

When cells of normal mesocotyls (epidermis) are compared with those in nana, the length of the normals is much the greater. The growth inhibition in nana therefore is a matter of decreased extension.

It has become clear in recent times that there is a hormone in plants which controls the longitudinal growth. This hormone is known as the *growth hormone, growth substance* or *auxin*. Without growth hormone no growth will occur, as F. W. Went (1928) proved.

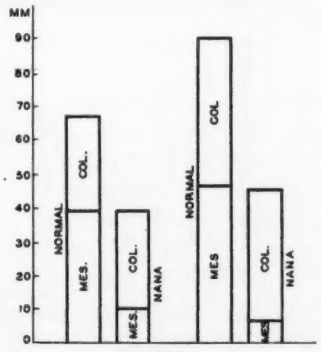

FIGURE 1

Length of 5-day (left-hand set) and 6-day old seedlings, showing the length of coleoptiles and mesocotyls separately.

2. *Less Auxin Given Off by Nana Than by Normal.*—In young corn seedlings the growth substance is produced in the tip of the coleoptile. If we cut off such tips and place them on blocks of 3% agar, the auxin will diffuse into the blocks. The auxin in the blocks can be analyzed by means of a test method designed by Went (1928) and further improved by Van der Wey (1931). From etiolated oat (Avena) seedlings the coleoptile tip is cut off and on the remaining stump a block of agar containing auxin is placed on one side. The auxin will diffuse out of the block and travel down on the side of the plant to which the block is applied. This causes a greater growth on this side, with the result that the stump will curve in a direction away from the block. The degree of curvature is directly proportional to the concentration of auxin in the block.

If a series of tips of nana and another series of normal are placed on agar blocks (see scheme Fig. 2), then, after a certain time, the agar blocks from the normals contain more auxin than those from the nanas. This is always the case, whether the tips are long or short (see table 4) or the time of extraction is longer or shorter. Even after replacing the same tips 7 times on new agar blocks, the nana always give off less growth substance than the normal tips under similar conditions. The average ratio can be seen from figure 2. This means that if the amount of auxin given off by nana is 10, then the amount given off by normal will be 18.

3. *Less Growth with a Given Amount of Auxin in Nana Than in Normal.*—From corn seedlings the tips were removed and the remaining coleoptile cut off and stuck into very wet sand. After 2 or 3 hours, when these coleoptile stumps had lost their auxin, a new cut surface was made at the apical end, and blocks of agar containing auxin were put on one side. From the amount of curvature conclusions can be drawn about the relative growth (see Van Overbeek, 1933, Fig. 24, p. 594). If the nana and normal coleoptiles are treated in this way, then the nanas appear to curve less than the normals. The ratio can be seen from figure 3, and is as 10 in nana to 18 in normal. This is the same ratio that we have seen for the inhibition of production of auxin, which suggests that there must be a close relation between the two phenomena. This was found to be due to a *higher destruction of auxin in nana, causing both the decrease in amount of auxin given off by nana tips and the smaller growth with a given amount of auxin in nana.*

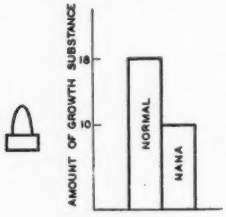

FIGURE 2

Relative amounts of growth substance given off by coleoptile tips of nana and normal. Average values of 400 tips each.

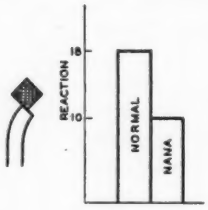

FIGURE 3

Relative curvatures with a given amount of growth substance applied one sidedly to nana and normal coleopti'e stumps. Average of 200 plants each.

4. *Destruction of Auxin Higher in Nana Than in Normal.*—The destruction of auxin can be demonstrated in the following way. If the inhibition of growth is considered in relation to the destruction of auxin a great destruction will be expected in those parts of plants with a greatly inhibited growth. One of these parts is the mesocotyl of nana. If cylinders cut out of these mesocotyls are placed with their *basal* cut surface on blocks of agar containing auxin, the results after one or more hours should show whether growth substance has disappeared from the blocks. Using cylinders of 4 or 5 mm. cut from the apical ends of mesocotyls of normal and nana, the resulting destruction is found to be much greater in nana. Hetero-auxin[2] and auxin are both destroyed. The possibility that the growth substance is not destroyed but merely transported against the direction of the polarity was disproved as follows. To the basal cut surface of mesocotyl cylinders 5 mm. long, 20,000 units of hetero-auxin was applied, and no growth substance could be found in blocks of plain agar at the apical cut surface.

These experiments show the destruction of auxin clearly, but they do not prove it to be the cause of the growth inhibition in nana. The basal

parts of normal mesocotyls appeared to destroy more auxin than the apical parts. Table 1 shows the destruction of auxin by mesocotyl cylinders 5 mm. long cut from the apical and basal part of the normals and from the apical part of nana.

TABLE 1

DESTRUCTION OF GROWTH SUBSTANCE BY MESOCOTYL CYLINDERS

EXP. NUMBER	AMOUNT OF GR.S. LEFT: NANA APICAL	NORMAL APICAL	NORMAL BASAL	AMOUNT OF GR.S. STARTED WITH		DURATION OF EXPERIMENT	AGE OF PLANTS
41026	4[1]	11[1]	2[1]	Auxin a	11	3 hours	5 days
41201	2	7	2	Hetero-	9	1	5
	2	7	..	auxin	9	1	4
41204	8	15	6	H-a	16		4

[1] All figures in the tables are mean values of 12 tips, cylinders, plants, etc.

To separate the results and causes of the growth inhibition, the destruction of growth substance in coleoptiles was investigated. As mentioned above the length of coleoptiles is the same in nana and normal. If there can be found a difference in destruction in coleoptiles, it certainly is *not* a *result* of a difference in growth. There was difficulty in getting coleoptiles and especially the tips free from auxin. This was found to be possible, however, by placing them for about 20 hours on very wet filter paper. After this time the tips do not give off auxin any more, but if placed on blocks of agar containing auxin they destroy it, as is shown in table 2.

TABLE 2

DESTRUCTION OF GROWTH SUBSTANCE BY COLEOPTILE TIPS OF 5 MM.

EXP. NUMBER	AMOUNT OF GR.S. LEFT. NANA	NORMAL	AMOUNT OF GR. S. STARTED WITH		DURATION OF EXPERIMENT	AGE OF PLANTS
50228	4.5	8.0	H-a	11.5	1 hour	5 days
50306	5.0	7.0	H-a	14.0	1¼	5
50308	3.6	6.0	H-a	7.5	1	5
	4.4	6.8	H-a	7.5	1	5
Total	17.5	27.8		40.5		
Destroyed	23	13				
Ratio	18	10				

The ratio of destruction in nana and normal tips is 10 to 18. We have seen before that the ratio of amount of auxin given off by these tips was 10 to 18. This makes it extremely likely that *the reduced amount of growth substance given off by nana can be explained by a higher destruction of growth substance in the dwarf.*

For coleoptile cylinders, cut from 5 to 10 mm. under the extreme tip, and freed of auxin, values were found as given in table 3.

These experiments show that in the region of nana coleoptiles which responds less to a given amount of auxin (see paragraph 3), the destruction of auxin is higher. The ratio of growth was 10 in nana to 18 in normal, the ratio of destruction is 16 in nana to 10 in normal.

From experiments in this paragraph, paragraph 8 and from experiments not yet published, it is clear that *the higher destruction of growth substance in nana is the cause of the dwarfed growth in nana.*

5. *In Coleoptiles Growth Substance Is Not the Limiting Factor.*—How is it, that in nana the length of the coleoptiles is the same as in normals,

TABLE 3

DESTRUCTION OF GROWTH SUBSTANCE BY COLEOPTILE CYLINDERS

EXP. NUMBER	AMOUNT OF GR.S. LEFT NANA	NORMAL	AMOUNT OF GR.S. STARTED WITH		DURATION OF EXPERIMENT	AGE OF PLANTS
50226	4.0	7.0	H-a	9.0	$3^1/_2$ hours	5 days
50306	5.7	8.4	H-a	14.0	$1^1/_4$	5
	7.0	9.3	H-a	14.0	$1^1/_4$	5
50308	2.5	4.0	H-a	7.5	1	5
	1.8	3.7	H-a	7.5	1	5
Total	21.0	32.4		52		
Destroyed	31	19.6				
Ratio	16	10				

though in nana coleoptiles more growth substance is destroyed than in normal ones? Growth is dependent not only on auxin, but on other factors too. The factor whose relative amount is least will limit the growth. Thus Went pointed out (1928), that in the basal parts of the coleoptile and mesocotyl of Avena, auxin is a limiting factor and therefore regulates the growth in these parts. In the upper parts of coleoptiles, however, another factor is limiting (cell extension material). Now the fact that nana destroys more auxin than normal and still has the same length of coleoptile can easily be explained by assuming that auxin is present in such large quantities that it is still not a limiting factor, though some of it is destroyed in nana. That an excess of auxin is present in the coleoptiles can be shown (table 4), which gives the amounts of auxin extracted from long and short coleoptile tips during 3 periods of $1^1/_2$ hours in succession. During the first period the tips 2 mm. long give off about the same amount as tips 20 mm. long. If these tips are replaced on new agar blocks, the long tips produce less than the short ones.

The growth of the mesocotyl is quite dependent on the amount of auxin left by the coleoptile. If this amount is large, a large growth of the mesocotyl will result; if the amount is small, then little mesocotyl growth will result. This can be shown very nicely by decapitation experiments of normal plants. The amount of auxin produced by a new "physiological tip," if a tip of about 1 cm. is cut off, is considerably less than the amount

produced by the original tip. Still this amount is sufficient for the coleoptile stump to grow as much as a comparable zone in the intact plant, but the growth of the mesocotyl is stopped completely. The amount of auxin left by the coleoptile is less in nana than in normal, because in nana more auxin is destroyed. The result will be a smaller amount of auxin available for the mesocotyl in nana, with a resulting smaller growth. This, together with the greater destruction of growth substance in the nana mesocotyl itself (paragraphs 4 and 6), is responsible for the smaller growth of the nana mesocotyls.

TABLE 4

AMOUNT OF GROWTH SUBSTANCE GIVEN OFF BY COLEOPTILE TIPS

Exp. number, 41024; plants, 5 days old; periods, $1^1/_2$ hours

	LENGTH OF TIPS	FIRST PERIOD	SECOND PERIOD	THIRD PERIOD
Nana	2 mm.	11.0	13.5	15.9
	20 mm.	11.6	9.0	9.8
Normal	2 mm.	18.6	17.1	19.0
	20 mm.	23.0	10.0	13.5

The *topmost internode* of corn plants grown under good conditions in the field is of the same length in normals and nanas (see photograph in Li, 1933). This can be explained in the same way as above for coleoptiles. It is a known fact that young flowers produce much growth substance. Very likely in young corn flowers so much auxin is produced that even if there is auxin destroyed in nana, the amount left is still not limiting for the growth in the internode below the flowers.

6. *Changed Oxidation-Reduction Properties in Nana?*—In what way is the auxin destroyed? Thimann (1934) tried to explain the fact that auxin in extracts is destroyed, by assuming that it is oxidized by enzymes. The enzymic activity of the two races of corn were therefore compared, and a higher catalase activity in nana was found (oxidase tests were negative). A higher catalase activity often indicates a higher peroxidase activity and changed oxidation-reduction properties. Cylinders of coleoptiles, mesocotyls and coleoptile tips of nana and normal were put into 5 cc. of a 0.03% pure H_2O_2 solution and shaken in an electric shaker in the dark room. After one hour 4 cc. of the solution was titrated with $KMnO_4$. In this way the amount of H_2O_2 destroyed could be determined. In coleoptile tips the catalase activity appeared to be higher in nana than in normal. In coleoptile cylinders and also in mesocotyl cylinders the same was found. A difference between tip and base in normal mesocotyls, however, could not be found. Table 5 gives some of the results.

TABLE 5

CATALASE ACTIVITY IN NANA AND NORMAL (AMOUNT OF H_2O_2 DESTROYED, IN 0.1 CC. OF $KMnO_4$)

In exp. 50122—15.5 cc. $KMnO_4$ = 4 cc. 0.03% H_2O_2
In exp. 50123—10.7 cc. $KMnO_4$ = 4 cc. 0.03% H_2O_2

EXP. NUMBER	COLEOPTILE TIPS		COLEOPTILE CYLINDERS		MESOCOTYL TIPS		MESOC. BASE	HOURS AFTER CUTTING OFF CYLINDERS OR TIPS
	NORMAL	NANA	NORMAL	NANA	NORMAL	NANA	NORMAL	
50122	38	43	..	..	83	110	75	0–1
	22	39	..	..	54	76	54	1–2
	16	30	..	..	43	67	43	2–3
	14	29	..	..	36	60	36	3–4
	9	19	..	..	27	44	22	4–5
50123	14	21	29	51	31	52	31	1–2
	7	10	13	30	19	30	19	3–4
Ratio	10	16	10	19	10	15		

The ratio of catalase activity in nana and normal is therefore between 15 and 19 in nana to 10 in normal, which certainly is of the same order as the ratios found for destruction of growth substance.

The *peroxidase*[3] activity was next compared in the two races. This was done by placing cylinders of nana or normal on agar blocks containing benzidine and H_2O_2. The more peroxidase, the more the benzidine is oxidized and the darker the color in the blocks. *The peroxidase activity appeared to be higher in nana than in normal*, and higher in the normal mesocotyl base than in the mesocotyl tip. In case of the higher peroxidase activity there are two possibilities: either the peroxidase oxidizes the auxin or it is merely an indicator of changed situations. If indeed the system peroxidase-peroxide is established, it is very likely that auxin is oxidized by it, since this system can oxidize compounds of the most varied types (see Onslow, 1921, p. 111). One thing seems possible, however, namely that in nana and normal the "oxidation levels" (rH?) are unlike.

7. *Making "Nana" from Normal by Heat Treatment.*—In one of his articles Von Euler (1919) describes how in yeast the catalase activity can be increased merely by raising the temperature to about 60°C. for one hour. If we can do the same with corn, and if an increase in catalase activity indicates a change in oxidation-reduction properties, and these in turn have something to do with the destruction of growth substance, we may expect normal plants after heat treatment to give plants of the type of nana. This was indeed found and not only this but the whole chain of differences as described above for nana and normal were found the same for treated and untreated normal plants. Normal plants were kept for one hour at a temperature of 45–50°C. in air. Six hours after treatment the *catalase activity* was 42 and 38 in treated coleoptile cylinders, against 23, 26 and 20 in untreated ones. Two hours after treatment the

amount of auxin given off by coleoptile tips was 3° in treated against 14° in the untreated ones. The *destruction* of auxin by mesocotyl tips was 3.5° in treated—$2^1/_2$ hour after treatment—against 0 in untreated. One day after treatment the treated destroyed 8° and the untreated only 1° from the 11° hetero-auxin started with. The *length* $1^1/_2$ days after treatment was 46 mm. in treated and 85 mm. in the untreated. *The phenomenon mentioned here certainly is an important support for the theory, that the dwarf type of growth of nana is due to an increased destruction of growth substance.*

8. *Summary.*—The causes of the dwarf type of growth of nana corn were studied. In nana more growth substance is destroyed than in normal, causing the inhibition of growth in the dwarf. The higher destruction of growth substance in turn may be due to a change in oxidation-reduction properties of nana.

[1] See Lindstrom (1923), Li (1933).

[2] Hetero-auxin appeared to be β-indolyl acetic acid, $C_{10}H_9O_2N$. Auxin a does not contain N, its formula is $C_{18}H_{32}O_5$ (see Kögl, 1935).

[3] *Catalase*, a haemin compound, reduces H_2O_2 to water and molecular oxygen. *Peroxidase*, also a haemin compound, reduces H_2O_2 and other peroxides only in presence of oxidizable compounds such as benzidine (and auxin). The oxygen from the H_2O_2 is "active" and carried from the peroxide toward the oxidizable substance. The optimum activity is found in a H_2O_2 concentration of 1:400.000.

Euler, H. von, and Blix, R., *Hoppe-Seyler*, **104,** 83 (1919).
Kögl, F., *Berichte deutschen chem. ges.*, **68,** Abt. A, 16 (1935).
Li, H. W., *Jour. Hered.*, **24,** 279 (1933).
Lindstrom, E. W., *Genetica*, **5,** 327 (1923).
Onslow, M. W., *Biochem. Jour.*, **15,** 107 (1921).
Thimann, K. V., *Jour. gen. phys.*, **18,** 23 (1934).
Van der Wey, H. G., *Proc. Amsterdam*, **34,** 875 (1931).
Van Overbeek, J., *Recueil des trav. bot. néerlandais*, **30,** 537 (1933).
Went, F. W., *Rec. des trav. bot. néerlandais*, **25,** 1 (1928).